Logic

Logic is a very important part of mathematics (as well as life itself, though I wouldn't say it's nearly as formal as it is in math), and you've heard me rant about things in it plenty of times. Why do we need logic though? Isn't math naturally logical, why should we even bring it up? Math is naturally logical, yes, but to be precise, it's only 'good math' that's logical.

Logic is based on evaluating the truth of statements or propositions. We represent these statements with propositional variables, usually P, Q, and R. In the logic we will be using, there are only two possible evaluations that a statement can have, true (1) or false (0). The precise way in which we evaluate the truth of a statement is through truth functions, which map the statements to their truth-ness. Generally we really won't think of them in this way, and instead just think of the variables P, Q, etc as already representative of the truth value.

1 Connectives

It wouldn't really be much of an interesting branch of mathematics if we just left it there though, so now we shall define the operators between propositional variables, called **connectives**.

Conjunction: This is the 'and' operator, represented by $P \wedge Q$. This is only true if both P and Q are true, and false otherwise.

$\wedge$	1	0
1	1	0
0	0	0

(This is a lot like a multiplication table (technically, this is called a Cayley Table), but there's really no reason why we can't do the same thing for another operation. Though not really important for this table, the first variable's values are in the first column, while the second variable's values are in the first row)

Disjunction: This is the 'or' operator, represented by $P \vee Q$. This is true if either P or Q are true, or both. It is false only then both are false.

$\vee$	1	0
1	1	1
0	1	0

Negation: This is the 'not' operator, represented by $\neg P$. This is true if P is false, and false if P is true. Essentially, it flips the argument.

$\neg$	1	0
	0	1

Conditional: This is the **material conditional** or **implication**, represented by $P \to Q$ (not to be confused with the function notation), and is a little more advanced than the previous three. So what exactly does this stand for though? The others are pretty self-apparent, representing rather intuitive concepts. The Conditional is essentially the 'if-then' statement. It is true if P and Q are true, but also true if P is false, whether or not Q is true or not. As an example, consider the following: "If I am a monkey, then the moon is cheese." Generally, I like to think that I'm not a monkey (debateable, I know), but the statement is still true nevertheless. However, if I were actually a monkey, then the statement would be false (of course, assuming the moon is not cheese). This statement is indicative of something important about the conditional. We did say that this was the 'if-then' statement, so wouldn't you think that the statements P and Q should actually be related? Certainly me being a monkey or not has nothing to do with with whether or not the moon is cheese. Nevertheless, the material conditional does not require any kind of actual causalty between the two things. Generally, this is seen as a problem (it would be really bad math to make up stupid theorems like "If this happens, then this totally unrelated things is true."), so we solve the problem with the **causal conditional**, represented the same way as the material one. From now on, we won't let the difference between the material and causal conditionals bother us at all, as it really doesn't matter here. Following with this sense of implication/if-then, the variable P is called the **hypothesis**, and the variable Q is called the **conclusion**.

$\rightarrow$	1	0
1	1	0
0	1	1

Biconditional: This is not much more than the conditional in both directions, also called **equivalence**, and represented by $P \leftrightarrow Q$. Earlier on, we had said things like "if and only if". Well, this is exactly what we meant there. P implies Q, and Q implies P. Hence, as long as the two truth values are the same, then the biconditional is true.

$\leftrightarrow$	1	0
1	1	0
0	0	1

So now we'll get into a few of the properties of these things. Conjunction and disjunction are rather simple; they are both associative and commutative, and the identities for them are truth and fallacy, respectively. The same applies to the biconditional (which has identity 1). As we can see by the Cayley Table, the conditional is not as lucky. We can also rewrite some of these connectives in terms of the others, as well. The conditional is equivalent to $(\neg P) \vee Q$, and the biconditional is equivalent to $(P \rightarrow Q) \wedge (Q \rightarrow P)$. How would we go about checking this, and in general finding the truth values of more complicated propositional statements? For this, we turn to **truth tables**.

Truth tables dissect a proposition into it's base components, and slowly adds on each statement until we have the whole statement. This is probably done most easily with an example; we will show that our alternative definition of the conditional is indeed true: we have $\neg P \vee Q$. Hence, we have P, Q, $\neg P$, and finally $\neg P \vee Q$. We shall create a column for each of these, and then each row will show the truth values that correspond:

P	Q	$\neg P$	$\neg P \vee Q$
1	1	0	1
1	0	0	0
0	1	1	1
0	0	1	1

In the end, the helpfulness of the truth tables are that they organize your thoughts, and allow you to have to only perform a single operation at a time.

When we talk about causal conditionals in mathematics, we usually want to know how things relate to eachother (it could be argued that that's really the only thing you do in mathematics). In the process of doing this, we want to know if something is a necessary condition for something, or sufficient. First, let's define what exactly we mean when we say these things. If something is a necessary condition, then it can't possibly be true if the necessary condition is not. The sufficient condition, on the other hand, is something that, if true, makes the original something irrevocably true.

Another thing we said earlier was that the conditional wasn't commutative (or associative), given $P \to Q$, the statement $Q \to P$ is completely different. This latter statement is called the **converse** of the former. On the other hand, the statement $\neg Q \to \neg P$ is equivalent to the former statement, while $\neg P \to \neg Q$ is equivalent to the latter. Let's make this a little neater:

$$(P \to Q) \leftrightarrow (\neg Q \to \neg P)$$

$$(Q \to P) \leftrightarrow (\neg P \to \neg Q)$$

Finally, we conclude by looking at the causal conditionals in a little more depth. Material conditionals are pretty straight-forward in creating: you literally just say that they're there, then boom! Causal conditionals are a little more difficult in creating, and the way is through proofs. There are several ways to go about making these proofs. Perhaps the easiest proof is for proving a statement wrong through the use of a counter-example. As an example, consider the proposition $n^2 = n + 2$. Now, obviously this isn't true, and a simple counter-example is $n = 0$, which gives us $0 = 2$, which is usually false (I say usually because when you get into modular arithmetic [the clock arithmetic I mentioned to you before]). Another type of proof is proof by contradiction. This works because we have $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$, which means that if we assume that the end result is false, and this shows that we have a contradiction (that is, $\neg Q \to \neg P$), then by our equivalence, the original proposition could be true. Of course, you could also make a direct proof, of which there are many other methods we won't discuss here.

2 Quantifiers

Quantifying basically means that we're saying which things make something true. As an example: for all husbands, there is a wife. The (first) quantifier in this case was "for all". We represent this with the symbol $\forall$, followed by whatever variables we're talking about (for our example, husbands). But note that we also established the existence of a wife, essentially quantifying her. This second quantifier, in the text given by 'is a', is "exists", represented by the symbol $\exists$, followed by whatever variables we're talking about (again, for the example, wife). Hence, if $husband = h$ and $wife = w$, then $\forall h \exists w$.