How to Become a Mathemagician: Mental Calculations and Math Magic

Adam Gleitman
(amgleit@mit.edu)

Splash 2012

"A mathematician is a conjurer who gives away his secrets."
– John H. Conway

This document describes the mathematical tricks that I have performed and goes into detail about the math behind them. First I provide how the trick appears from the perspective of the audience, and then I provide an explanation.

1 Squaring Numbers

You ask a member of the audience for a two-digit or three-digit number. After a few seconds, you say what the square of that number is, without the aid of a calculator.

1.1 The Explanation

Let N be the number you are given. The trick is you want to find two numbers a and b that add up to N such that it is easy to compute a^2, b^2, and $2ab$.

The reason behind this trick is the well-known fact $(a+b)^2 = a^2 + 2ab + b^2$. By rewriting it in this style, you can make it easier to do the calculations in your head. For example, suppose $N = 61$. Then you could choose $a = 60$ and $b = 1$, and your answer will be $60^2 + 2(60)(1) + 1^2 = 3600 + 120 + 1 = 3721$. This is one of the essential themes behind this class: *Try to turn a hard problem into several easy ones.*

For three-digit numbers, the same concept applies, although you may have to break it down into several stages. For example, suppose $N = 354$. A good first step would be to take $a = 300$ and $b = 54$. Then $a^2 = 90000$ and $2ab = 300 \cdot 108 = 32400$. To calculate 54^2, rewrite 54 as $50 + 4$, which yields $50^2 + 2(50)(4) + 4^2 = 2916$. Adding these three numbers together yields a final answer of 125316.

Occasionally it may help to make one of a or b a negative number. For example, suppose $N = 97$. Then a good choice would be $a = 100$ and $b = -3$. This yields a final answer of $100^2 - 2(100)(3) + (3)^2 = 10000 - 600 + 9 = 9409$.

1.2 Useful Tips

- If $N \approx 50$, try setting $a = 50$. Then $2ab = 100b$. If b is a single-digit number, a convenient shortcut would be to take $25 + b$ and append b^2 as a two-digit number. For example, $52^2 = \underline{27}\,\underline{04}$ and $43^2 = \underline{18}\,\underline{49}$.
- If $N \approx 100$, try setting $a = 100$. Then $2ab = 200b$. If b is a single-digit number, a convenient shortcut would be to take $100 + 2b$ and append b^2 as a two-digit number. For example, $108^2 = \underline{116}\,\underline{64}$ and $99^2 = \underline{98}\,\underline{01}$.
- If N ends in a 5, one strategy is to drop the trailing 5, multiply the rest of the number by 1 more than itself, and append 25. For example, $65^2 = \underline{42}\,\underline{25}$ and $115^2 = \underline{132}\,\underline{25}$.
- If $N \approx 500$, try setting $a = 500$. Then $2ab = 1000b$. If b is small enough, a convenient shortcut would be to take $250 + b$ and append b^2 as a three-digit number. For example, $513^2 = \underline{263}\,\underline{169}$ and $491^2 = \underline{241}\,\underline{081}$. This still sort of works even if $b^2 \geq 1000$, but you have to make sure you adjust the $250 + b$ part. For example, $533^2 = 283000 + 1089 = 284089$.

- If $N \approx 1000$, try setting $a = 1000$. Then $2ab = 2000b$. The same strategies for when b is small enough still apply here.
- It's always pretty safe to set a to represent the first digit and b the remaining digits. In the case when N is a three-digit number, you can keep track of $a^2 + 2ab$ while working out b^2.
- Practice, practice, practice. After a while you can start to work out which patterns work out well, and you will even start being able to memorize some two-digit squares!
- Memorize $69^2 = 4761$. If anyone says 69, you can then say, "I memorized that one because I knew that somebody would say it."

2 Day of the Week

You ask the audience, "Who here knows what day of the week they were born on?" After a show of hands, you call on someone and ask for their date of birth. That person says, "November 15, 1996." After thinking for a few seconds, you correctly identify that day as a Friday.

2.1 The Explanation

There are many ways to do this trick. One way to do it is known as the *Doomsday algorithm*, invented by mathematician John Conway. His method will allow you to identify any day of the week given a date on the Gregorian calendar, even one many years in the past or in the future. It can be broken down into three basic steps:

- Determine the "base day" for the century.
- Use the base day to calculate the "doomsday" for a particular year.
- Choose a reference date out of a list of dates that are always doomsdays, and use that as a reference point.

Here is a description of the basic steps you need to do:

2.1.1 Determine the Base Day

To calculate the base day for a particular century, use the following table:

Century	Base Day
1700s	Sunday
1800s	Friday
1900s	Wednesday
2000s	Tuesday
2100s	Sunday

The base days are periodic every four centuries.

2.1.2 Calculate the "Doomsday"

Once you have the base day for your century, look at the last two digits of the year. Call it y. Then the doomsday for your year is the base day plus $y + \lfloor \frac{y}{4} \rfloor$ days, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x (alternatively, x rounded down to the nearest integer). For example, if the year in question is 1960, then doomsday will be $60 + \lfloor \frac{60}{4} \rfloor$, or 75, days after Wednesday, which is a Monday. (The $\lfloor \frac{y}{4} \rfloor$ helps take leap years into account.)

One tip here is that you can subtract 28 from y without changing the answer. You'd be skipping over 35 days, which is exactly 5 weeks, so you wouldn't be changing the day of the week.

2.1.3 Finish the Job

From here, you want to choose a reference date that will always be doomsday in a given year. Luckily, these dates are not difficult to memorize. Here is a table of months and reference dates you can remember:

Month	Reference Day
January	1/3 on non-leap years, 1/4 on leap years
February	The last day of February
March	3/7
April	4/4
May	5/9
June	6/6
July	7/11
August	8/8
September	9/5
October	10/10
November	11/7
December	12/12

To help memorize the rule for January, note that leap years generally occur once every **4** years, while the other **3** years are not. Months April through December are rather easy to memorize. The even-numbered months are 4/4, 6/6, 8/8, 10/10, and 12/12. The odd-numbered months are 5/9, 9/5, 7/11, and 11/7. You can use the mnemonic "a 9-to-5 job at 7-11."

Each of these days will be a doomsday in the given year. For example, November 7, 1960 was a Monday. From here, all you need to do is count forward or backward.

3 In Reverse

You ask the audience for a three-digit number where the digits are in strictly decreasing order, and tell them to follow these steps:

1. Reverse the digits of your number and subtract your new number from the original number.

2. Reverse the digits of the new number and add this number to the difference.

You then open a sealed envelope that contains their final number, which you predicted before the show!

3.1 The Explanation

The secret is rather simple:

Theorem 1. *The answer will always be 1089.*

Proof. Let a, b, and c be the digits of the original number. Then the original number is of the form $100a + 10b + c$, and when you reverse the digits you get $100c + 10b + a$. Subtracting these two, we get $99a - 99c = 99(a - c)$. Since $a > b > c$, we may conclude that $2 \leq a - c \leq 9$, and that the only possible differences we can get are 198, 297, 396, 495, 594, 693, 792, and 891.

Each of these numbers can be represented in the form $100k + 90 + (9 - k)$. Reversing the digits of this number yields $100(9-k)+90+k$. If we add these together, the k's cancel out and we are left with $900+180+9 = 1089$. $\square$

For example, if the original number is 843, reversing and subtracting yields $843 - 348 = 495$. Reversing again yields 594, and $495 + 594 = 1089$.

3.2 Generalizing

You might wonder if this trick works for 4-digit numbers or larger. In this case, we will relax the constraint that the digits must be in strictly decreasing order to a lighter constraint: the difference must have the same number of digits as the original number.

It turns out that when you start with a 4-digit number or larger, the final number won't always be the same. However, the list of numbers you can end up with is still rather compact:

Number of digits	Possibilities
3	1089
4	9999; 10,890; 10,989
5	99,099; 109,890; 109,989
6	8 possibilities
7	8 possibilities
8	21 possibilities

I invite you to try to figure out a pattern in how many possible final numbers there are if you start with a k-digit number. Here's a hint: for the different results you can get, look at when you have to "carry" in your initial subtraction.

4 Add 'Em Up

You ask a volunteer to write down two small numbers of his choice on a piece of paper. You then ask him to continue writing numbers such that each number is the sum of the previous two until he gets bored. He hands you the list. You stare at it for a few seconds and recite the sum of all the numbers on the list.

4.1 The Explanation

Let's work through an example and see if we can find a pattern. Define a sequence $a_1 = 3$, $a_2 = 4$, and for all $n > 2$, $a_n = a_{n-1} + a_{n-2}$. Furthermore, define S_n to be the sum of the first n numbers in this sequence.

n	1	2	3	4	5	6	7	8	9	10	11	12
a_n	3	4	7	11	18	29	47	76	123	199	322	521
S_n	3	7	14	25	43	72	119	195	318	517	839	1360

If you look carefully, you will notice that $S_n = a_{n+2} - 4$. This is indeed true, and we will prove this using a technique called *induction*. The idea is that we will prove this formula for $n = 1$, and then show that if it is true for a particular value of n, then it is also true for $n + 1$. Think of induction like a chain of dominos. You push the first domino over, and if one domino falls, the next domino will fall. Therefore all the dominos will fall down. We actually generalize this for general starting values:

Theorem 2. *Let $\{a_k\}$ be a sequence such that for all $k > 2$, $a_k = a_{k-1} + a_{k-2}$. Then for any $n \geq 1$,*

$$\sum_{i=1}^{n} a_i = a_{n+2} - a_2.$$

Proof. By induction.
Base case:
$n = 1$. Then our formula is equivalent to $a_1 = a_3 - a_2$, which is true by our recursive definition.
Inductive step:
Suppose that, for a particular value of n, $\sum_{i=1}^{n} a_i = a_{n+2} - a_2$. If we add a_{n+1} to both sides we get:

$$\sum_{i=1}^{n+1} a_i = a_{n+1} + a_{n+2} - a_2 = a_{n+3} - a_2$$

So if our formula works for one value of n, it works for the next value, and the induction is complete. □

This sea of formality may seem a bit intimidating. Here's a more intuitive way to grasp this. Note that $a_3 = a_1 + a_2$ by our recursive definition. We can rearrange this to $a_1 = a_3 - a_2$. Now we'll add various terms:

$$\begin{aligned}
a_1 &= a_3 - a_2 \\
a_2 + a_1 &= a_2 + a_3 - a_2 \\
&= a_4 - a_2 \\
a_3 + a_2 + a_1 &= a_3 + a_4 - a_2 \\
&= a_5 - a_2 \\
a_4 + a_3 + a_2 + a_1 &= a_4 + a_5 - a_2 \\
&= a_6 - a_2
\end{aligned}$$

This should give you a better idea of what the inductive process is. By adding consecutive terms, you can see that our formula still holds as you move on.

However, there's one more thing you have to cover: How do we calculate a_{n+2} if we are only given the numbers a_1 through a_n? Here is an easy way to do it:

$$\begin{aligned}
a_{n+2} &= a_n + a_{n+1} \\
&= a_n + a_n + a_{n-1} \\
&= 2a_n + a_{n-1}
\end{aligned}$$

This gives you an easy way to compute a_{n+2} in terms of numbers already written down. In fact, you can even cover up everything except the first two and last two numbers and still be able to get the sum efficiently!

5 Five-Card Monte

You grab a deck of cards. You ask five volunteers to each cut the deck. You then deal out the top five cards in order, one to each volunteer. You ask the people with the red cards to step forward and mentally communicate their card. After a few seconds, you name off each of the cards that your five volunteers are holding.

5.1 The Explanation

There are two things special about the deck of cards you are using. First of all, you're not using a full deck; you're only using 32 cards: aces through eights of each suit. Second, the cards have been arranged in a certain order beforehand:

8♣, A♣, 2♣, 4♣, A♠, 2♢, 5♣, 3♠, 6♢, 4♠, A♡, 3♢, 7♣, 7♠, 7♡, 6♡,
4♡, 8♡, A♢, 3♣, 6♣, 5♠, 3♡, 7♢, 6♠, 5♡, 2♡, 5♢, 2♠, 4♢, 8♠, 8♢

Note that cutting the deck preserves this order, so it does not interfere with the trick.

Any of the 32 possible sequence of 5 red and black cards appear consecutively in this sequence exactly once. For example, you can find black-black-red-red-black starting at the 6♣. You are allowed to wrap around the end of the sequence; for example, you can find black-red-black-black-black by starting at the 8♠ and continuing with 8♢, 8♣, A♣, and 2♣.

5.2 A Lock Problem

To set the foundations for the in-depth explanation of this problem, consider the following problem. Suppose you have a lock with two buttons on it: 0 and 1. To open the lock, you need to enter a three-digit combination, like 011. The lock will open as long as the last three buttons pushed are the combination. For example, if you enter 10011, the lock will open. You want to find a sequence of 0s and 1s such that if you press those buttons, you are guaranteed to eventually open the lock (you're allowed to check whether the lock is opened at any time). What is the shortest sequence that will do this?

We can establish an upper bound of 24 with the following strategy: Enter each code, and test the lock after each entry. Your sequence would then be:

000 001 010 011 100 101 110 111

However, this would be rather wasteful. You don't need to press those two zeros to test 001; you can just press 1 and test the 001 combination just then. Ideally, we'd like a sequence for which every button press after the first three will yield a new, untested combination. Such a sequence would only be 10 digits long, and it is the best that we can do.

In this case, it is possible to construct such a sequence:

0001011100

Every three-digit combination appears exactly once. If we drop the last two digits, we get:

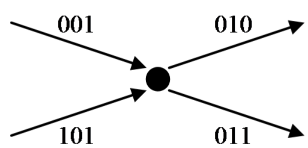
00010111

Then every three-digit combination appears exactly once if we allow wrapping. This is called a *de Bruijn sequence*. Here's a more general definition:

Definition 3. *A de Bruijn sequence of order k is a sequence of 2^k 0s and 1s such that every possible k-digit sequence of 0s and 1s appears exactly once if you allow wrapping.*

5.3 Do De Bruijn Sequences Always Exist?

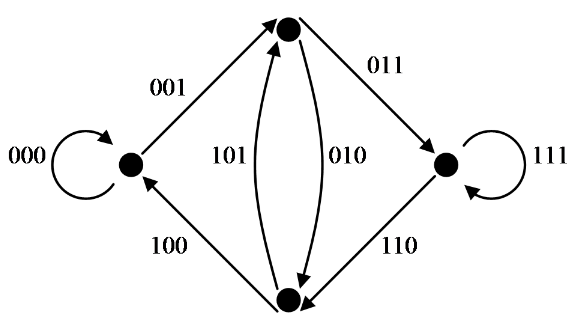
In this section we will model de Bruijn sequences of order k as a network of 2^k arrows known as a *directed graph*. Each arrow, or *edge*, will represent a single k-digit combination. These arrows will form paths corresponding to a series of button presses in the lock problem. For example, arrow 001 will lead into arrows 010 and 011, because these combinations can be checked after 001 by pressing either 0 or 1, respectively. Similarly, arrow 101 will also lead into arrows 010 and 011.



We will refer to the place where the arrows meet as a *vertex* or *node*.

Similarly, arrow 010 will lead to arrows 100 and 101. However, in the above diagram we already used arrow 101. What do we do? Our solution is to extend the tail of the 101 arrow into the head of the 010 arrow. In other words, these arrows should lead into each other.

If you do this for all eight arrows, you make the following network:



A de Bruijn sequence will then correspond to a path through this graph that passes through every edge exactly once and that starts and ends at the same node. In general, such a path is called a *Eulerian cycle*, named after Leonhard Euler. Luckily, there is a condition for connected (meaning that there is a path between any two nodes) directed graphs to meet that guarantees the existence of a Eulerian cycle.

Theorem 4. *A connected directed graph has a Eulerian cycle if and only if every node v is such that the number of edges going into v equals the number of edges coming out of v.*

Proof. First we prove that if a Eulerian cycle exists, then every node has the same number of edges going into it as it does coming out of it. We prove this by virtue of the fact that a Eulerian cycle starts and ends at the same vertex. Suppose our Eulerian cycle passes through vertices $v_1, v_2, \ldots, v_n$ before returning back to v_1. (Note that it is possible for two different v_i's to refer to the same vertex.) Every time a vertex appears in this list, we can account for one edge that enters it and another edge that leaves it. Therefore the number of edges that flow into any particular node must equal the number of edges that flow out of it.

Now we prove that if every node has the same number of edges going into it as it does coming out of it, then the graph has a Eulerian cycle. This is more involved. Let P be the longest path in the graph that does not go over any edge more than once, and suppose it visits nodes $v_0, v_1, v_2, \ldots, v_n$, in that order.

We first claim that $v_n = v_0$, which we prove by contradiction. Assume that $v_n \neq v_0$. Then all the edges that come out of v_n must already be in P; otherwise we could construct a longer path. If v_n appears elsewhere in the path, then we account for an edge flowing into v_n and one flowing out of v_n. Therefore the edge (v_{n-1}, v_n) gives us a stray edge flowing into v_n with no edge flowing out to counterbalance it. This contradicts the fact that the number of edges flowing in equals the number of edges flowing out, so we must have $v_n = v_0$. In other words, P is a cycle (but we don't know that it's a *Eulerian* cycle... at least, not yet).

Now assume for the purposes of contradiction that P is not a Eulerian cycle. Then there is some edge (u_1, u_2) that is in the graph but not in P. Let v be an arbitrary vertex in P. Since the graph is connected, there exists some path P' from u_2 to v. Concatenating (u_1, u_2), P', and P gives a path that is longer than P, which means that P can't be the longest path. This is our contradiction. Therefore P is a Eulerian cycle. □

Every graph that represents a de Bruijn sequence is such that every node has two edges flowing out of it and two edges flowing into it. Therefore we have the following corollary:

Corollary 5. *For any $k \geq 1$, there exists a de Bruijn sequence of order k.*

5.4 Back to the Card Trick

If we treat our sequence of cards as a sequence of 1s and 0s, with 1s representing red cards and 0s representing black cards, then we get the following de Bruijn sequence of order 5:

00000100101100111110001101110101

We have chosen our card sequence such that it is easy to go from binary digits to a card. If we have digits $ABCDE$, then the card corresponding to the position of A can be encoded in the following way: AB indicates the suit, and CDE indicates the rank:

AB	suit
00	clubs
01	spades
10	diamonds
11	hearts

CDE	rank	CDE	rank
000	8	100	4
001	A	101	5
010	2	110	6
011	3	111	7

A is a 1 if the card is red and 0 if the card is black. B is 1 if the card is shaped like a heart (or a major suit for you bridge players) and 0 if not. CDE represents the rank in a binary encoding, with 0 mapping to 8.

For example, 10011 maps to 3♢, and 00110 maps to 6♣.

This just leaves one issue: we don't want to memorize an entire sequence of 32 numbers! Luckily, this sequence contains a pattern. If $ABCDEF$ are six consecutive digits in this de Bruijn sequence, then $F = A \oplus C$ (most of the time), where $\oplus$ is defined by the following table:

$\oplus$	0	1
0	0	1
1	1	0

The only exceptions are when $ABCDE$ is 10000 or 00000, in which case the opposite is true. If we don't make this exception, then either our sequence will not contain 00000, or it will consist entirely of 0s. This minor edge case can slip you up if you don't remember it (and it has slipped me up when practicing this trick!), so the best advice I can give you is to practice.

5.5 Some Final Words about This Trick

You can actually apply some real sleight-of-hand to this trick. If you want to be more convincing, you can perform what looks like a shuffle, but what is really a cut.

Finally, don't tell anyone that the deck of cards you're using is not a full deck! If you don't bring it up, your audience may not even notice!

6 The H4xx0r's Trick

You give a volunteer the following instructions:

1. Think of any whole number.
2. Multiply it by 1 more than itself.
3. Add 1337. (This is why I call it "The H4xx0r's Trick!")
4. Square the result.
5. Divide the result by 24.

You then ask the volunteer to name everything before the decimal point, and you jump in and say everything after the decimal point!

6.1 The Explanation

Since you only need to identify what's after the decimal point, you just need to know the remainder when your volunteer's number is divided by 24. If you work out the math, your volunteer will end up with $(n(n+1)+1337)^2$ at the end of step 4. What is the remainder when we divide this by 24?

To help us with this problem, we can use *modular arithmetic*. The idea of modular arithmetic is that if two numbers a and b leave the same remainder when you divide them both by some number m, they are essentially the same. This is written as $a \equiv b \pmod{m}$. An alternative, more rigorous definition of this is saying that $a - b$ is a multiple of m.

When performing addition, subtraction, or multiplication $\bmod m$, we can safely replace any number with another number that is congruent to it $\bmod m$.

We will show that the remainder is always the same, but first we'll need a few lemmas, or stepping stones in a more complex proof:

Lemma 6. *For all integers n, $(n(n+1)+1337)^2 \equiv 1 \pmod{3}$.*

Proof. We prove this by casework. First note that $1337 \equiv 2 \pmod{3}$.

If $n \equiv 0 \pmod{3}$, then:

$$\begin{aligned}(n(n+1)+1337)^2 &\equiv (0(0+1)+2)^2\\ &\equiv 2^2\\ &\equiv 4\\ &\equiv 1 \pmod{3}\end{aligned}$$

If $n \equiv 1 \pmod 3$, then:

$$\begin{aligned}(n(n+1)+1337)^2 &\equiv (1(1+1)+2)^2 \\ &\equiv 4^2 \\ &\equiv 16 \\ &\equiv 1 \pmod 3\end{aligned}$$

If $n \equiv 2 \pmod 3$, then:

$$\begin{aligned}(n(n+1)+1337)^2 &\equiv (2(2+1)+2)^2 \\ &\equiv 8^2 \\ &\equiv 64 \\ &\equiv 1 \pmod 3\end{aligned}$$

We've exhausted all possible cases, and in every single one, our expression is $\equiv 1 \pmod 3$. ☐

Lemma 7. *For all integers n, $(n(n+1)+1337)^2 \equiv 1 \pmod 8$.*

Proof. We can actually be a bit more slick here than just straight-up casework.

Since n and $n+1$ are consecutive integers, at least one of them must be even. Therefore $n(n+1)$ is even, so $n(n+1)+1337$ must be odd. We can express it in the form $2k+1$ for some k.

Note that $(2k+1)^2 = 4k^2+4k+1 = 4k(k+1)+1$. Like before, $k(k+1)$ must be even, and it is of the form $2t$. Then $(2k+1)^2 = 8t+1$, which means it always leaves a remainder of 1 when we divide it by 8. ☐

Now, let's get on with the main theorem:

Theorem 8. *For all integers n, $(n(n+1)+1337)^2 \equiv 1 \pmod{24}$.*

Proof. Let $N = (n(n+1)+1337)^2$. By our lemmas, we know that $N \equiv 1 \pmod 3$ and $N \equiv 1 \pmod 8$. Then $N-1$ must be both a multiple of 3 and a multiple of 8. So $N-1$ is a multiple of 24, and $N \equiv 1 \pmod{24}$. ☐

From here, all you need to know is that the decimal expansion of $\frac{1}{24}$ is $0.041\overline{6}$.

7 Money Talks

You pull out some coins from your pocket: pennies, nickels, dimes, and quarters. You call up four volunteers and ask each of them to take a single coin from the pile while you turn your back, asking that they each take a different denomination

You then tell your volunteers to then take some additional money from the pile. You tell the first person, "Take as much money from the pile as you took before. You can take more than one coin if you want, and you can make change with the pile as much as you want." You tell the second person, "Take twice as much money as you did before. For example, if you took the dime, take another 20 cents." The third person takes three times as much, and the fourth person takes four times as much, all while your back is still turned. You tell them they can make change with each other or with the pile as much as they like.

You scoop the remaining coins into your hand, shake them a bit, and you correctly identify who took which coin.

7.1 The Explanation

You can use modular arithmetic to deduce who took which coins.

The money in the pile is of a certain distrubtion: six pennies, six nickels, six dimes, and four quarters, for a total of 196 cents. Once the volunteers each take their first coin, they will have taken $1+5+10+25 = 41$ cents, leaving 155 cents left in the pile.

Let p, n, d, and q represent the volunteer numbers that took the penny, nickel, dime, and quarter respectively. Each of these will be a distinct number between 1 and 4 inclusive. Then the volunteers will take an additional

$p + 5n + 10d + 25q$ cents from the pile during the second stage. Let $T = 155 - p - 5n - 10d - 25q$, the number of cents left.

You should discreetly count the money as you pick it up from the table. You can do it by glancing or by picking up the coins a few at a time. This way you'll know the value of T, and you can use it to calculate which volunteer took which coin in the first stage.

These derivations may not be obvious at first, and actually getting to them most likely took a good amount of thinking. Some of these, like figuring out who took the penny, are fairly straightforward. Others, like figuring out who took the dime, are not as obvious. Problems like this may seem intimidating at first, but those of you who have done math competitions probably know the golden rule: TRY SOMETHING!

"It is common sense to take a method and try it. If it fails, admit it frankly and try another. But above all, try something."

– Franklin D. Roosevelt

7.1.1 Who Took the Penny?

To figure out who took the penny, reduce our T equation mod 5:

$$\begin{aligned} T &\equiv -p \\ &\equiv 5 - p \pmod 5 \\ p &\equiv 5 - T \pmod 5 \end{aligned}$$

That's how you compute p.

7.1.2 Who Took the Dime?

To figure out who took the dime, first we reduce our T equation mod 4:

$$T \equiv 3 - p - n - 2d - q \pmod 4$$

It may seem like we're stuck. However, we can rewrite the above equation like this:

$$T \equiv 3 - d - (p + n + d + q) \pmod 4$$

Note that p, n, d, and q are each different numbers from 1 to 4. Therefore $p + n + d + q = 10$, so we can continue:

$$\begin{aligned} T &\equiv -7 - d \\ &\equiv 5 - d \pmod 4 \\ d &\equiv 5 - T \pmod 4 \end{aligned}$$

That's how you compute d.

7.1.3 Who Took the Nickel?

First, let's divide our T equation by 5:

$$\frac{T}{5} = 31 - \frac{p}{5} - n - 2d - 5q$$

Now we will round both sides down to the nearest integer. Note that since $1 \leq p \leq 4$, we have $30.2 \leq 30 - \frac{p}{5} \leq 30.8$. Therefore rounding this down to 30 will be sufficient. We get:

$$\left\lfloor \frac{T}{5} \right\rfloor = 30 - n - 2d - 5q$$

Reducing this mod 5 yields:

$$\left\lfloor \frac{T}{5} \right\rfloor \equiv -n - 2d \pmod 5$$
$$n \equiv -\left\lfloor \frac{T}{5} \right\rfloor - 2d \pmod 5$$

This may seem unsettling, but you already calculated what d is, so this isn't much of a problem. Just calculate $\left\lfloor \frac{T}{5} \right\rfloor + 2d$, and subtract this value from the next highest multiple of 5. Now you know what n is.

7.1.4 Who Took the Quarter?

You know who took the penny, who took the nickel, and who took the dime. Therefore, the remaining volunteer must have taken the quarter.

7.2 An Example

Suppose you find 66 cents left. Then your thought process would be:

1. $66 \equiv 1 \pmod 5$, and $5 - 1 = 4$, so volunteer #4 took the penny.
2. $66 \equiv 2 \pmod 4$, and $5 - 2 = 3$, so volunteer #3 took the dime.
3. $\frac{66}{5}$ rounded down is 13. Volunteer 3 took the dime, and $13 + 2 \cdot 3 = 19$, and $20 - 19 = 1$. Therefore spectator #1 took the nickel.
4. The remaining volunteer, volunteer #2, must have taken the quarter.

You can reveal these in any order you want.

8 Three, Five, Seven

You ask a volunteer to think of a number between 1 and 100. You ask for the remainders when the number is divided by 3, by 5, and by 7. After a few seconds, you identify the number.

8.1 The Explanation

The secret that makes this trick work is the Chinese Remainder Theorem, which states that as long as no two divisors share any common factor other than 1, there will always be a solution.

For each divisor d, you want to find a number that leaves a remainder of 1 when you divide it by d but leaves no remainder when you divide it by any other divisor. In this case, such numbers are 70, 21, and 15. Then we have the following theorem:

Theorem 9. *Let N be an integer, and let a, b, and c be the remainders when N is divided by 3, 5, and 7, respectively. Then $N \equiv 70a + 21b + 15c \pmod{105}$.*

Proof. We will prove that they are congruent to each other mod 3, mod 5, and mod 7, similar to how we proved the theorem in trick 6.

If you reduce $N \equiv 70a + 21b + 15c$ in mod 3, mod 5, and mod 7, you get $a \equiv a \pmod 3$, $b \equiv b \pmod 5$, and $c \equiv c \pmod 7$, respectively, each of which is obviously true. Therefore $N \equiv 70a + 21b + 15c$ in mod 3, mod 5, and mod 7.

In other words, $N - (70a + 21b + 15c)$ is a multiple of 3, a multiple of 5, and a multiple of 7, so it must be a multiple of LCM$(3, 5, 7) = 105$, so $N \equiv 70a + 21b + 15c \pmod{105}$. □

For example, if you are given $a = 2$, $b = 2$, and $c = 1$, then you know that $N \equiv 70 \cdot 2 + 21 \cdot 2 + 15 \cdot 1 = 197 \pmod{105}$. To get N between 1 and 100, just subtract a multiple of 105 to get 92.

One tip for this trick is that if a, b, and c are all greater than 1, you can try subtracting a constant from each of a, b, and c to get a number that might be easier to work with. In the above example, you could find that $N - 1$ leaves remainders of 1, 1, and 0, so $N - 1 = 70 + 21 = 91$, or $N = 92$. This can help speed up the mental calculations.

9 Circle One Digit

You give a volunteer the following instructions as you look away:

1. Think of any four-digit number.
2. Form another number by mixing up the digits of your first number.
3. Subtract the smaller from the larger.
4. Multiply the difference by any number you want, preferably a large number. Write the product down.
5. Choose any *nonzero* digit and circle it.
6. Name off all the other digits in any order.

After a second, you identify which digit the volunteer circled!

9.1 The Explanation

This is one of my favorite tricks because it appears amazing but the explanation is very simple: *The sum of the digits of the final number will always be a multiple of 9.*

First we will start with another fact: any number is congruent to the sum of its digits mod 9. We will show this for four-digit numbers. Suppose you have a four-digit number $ABCD$. Then you can express this number as $1000A + 100B + 10C + D$. The sum of its digits is obviously $A + B + C + D$. Subtracting one from the other yields $999A + 99B + 9C$, which is clearly a multiple of 9.

If X is the first number and Y is the number formed by mixing up the digits of X, then X and Y have the same digit sum s. Then we have $X \equiv s \pmod 9$ and $Y \equiv s \pmod 9$, so $X \equiv Y \pmod 9$, so $|X - Y|$ is a multiple of 9. Multiplying this difference by any other number will maintain the property that it's a multiple of 9, so its digit sum will also be a multiple of 9.

This is how you can figure out the remaining digit: figure out what number you need to make the sum of the digits in the final number a multiple of 9. This also helps explain why you don't want the volunteer circling a zero: you won't be able to tell if it's a zero or a nine, since both of them will work!

"You are the master. The math is the slave. Never let math enslave you."
– Donald R. Sadoway