Topics in Mathematics

Pythagorean Theorem Lecture Summary (Written and Presented by Andrew Spieker)

Hopefully you all know the famous Pythagorean Theorem, which relates the sides of a right triangle in the following way:

If I label the “legs” of the right triangle as “a” and “b,” and the “hypotenuse” as “c,” then:

$$a^{2}+b^{2}=c^{2}$$

A nice little fact: of all the theorems in the world, this is the one to which we have the most proofs. This particular webpage gives about 80 different proofs of it:

<http://www.cut-the-knot.org/pythagoras/index.shtml>

**Then, we completed the Pythagorean Theorem Proof Exercise; the answers are posted in our documents.**

We proved that if $(a,b,c)$ is a primitive Pythagorean Triple (PPT,) then so is $(la,lb,lc)$. Here is the proof:

 Suppose $\left(a,b,c\right)$ is a PPT. Then $a^{2}+b^{2}=c^{2}$

 Multiply the equation through by $l^{2}$ and get that $\left(l^{2}a^{2}\right)+\left(l^{2}b^{2}\right)=(l^{2}c^{2})$

 We rewrite that as $\left(la\right)^{2}+\left(lb\right)^{2}=\left(lc\right)^{2}$

 Therefore $(la,lb,lc)$ is a Pythagorean Triple.

Then, we came up with a *generating* function for Pythagorean Triples:

 We note that if we have a PPT, $(a,b,c)$, that $a^{2}+b^{2}=c^{2}$, or $a^{2}=c^{2}-b^{2}$

 We factor the right hand side, and get that $a^{2}=\left(c-b\right)(c+b)$. Since $(a,b,c)$ is a PPT, that means that $a,b,c$ have NO common divisors, or they are *coprime*. Therefore, $c+b$ and $c-b$ have no common divisors. BY THE WAY: I didn’t prove this in class, because my mind was drawing a blank; here is the proof:

Proof: Suppose $c+b$ and $c-b$ have common factors. This means that for some integer $d$, $d$ divides $c+b$ and $d$ divides $c-b$. If this is the case, then $d$ divides $2c$ (by adding the equations), and $d$ divides $2b$ (by subtracting the equations). Then, this implies that $b$ and $c$ have a common factor, which we sais wasn’t true. So, therefore, our original assumption that $c+b$ and $c-b$ have common factors was not true. Therefore, $c+b$ and $c-b$ have NO common factors!

 The only way that two numbers without common divisors could ever be a perfect square is if they were both perfect squares to begin with. Why is this true? See if you can prove this one on your own ☺

 Then, we rewrite $c-b$ as $s^{2}$ and $c+b$ as $t^{2}$ with $s,t$ coprime and $1\leq s<t$. We solve this equation for $a,b$ and $c$ and we get the following function that generates EVERY SINGLE Primative Pythagorean Triple:

$$a=st, b=\frac{s^{2}-t^{2}}{2}, c=\frac{s^{2}+t^{2}}{2}$$