Notes On Topology
Day 1
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Definition 1.1: For some set, A and a function, f, defined on a domain $B \supset A$, we say that A **is closed under** f if $\forall x \in A$, $f(x) \in A$.

Example 1.2: Take the set of even intergers and the function f defined on Z (the integers) sending $x \to -x$. The even integers are closed under f.

Definition 1.3: A **function** is an ordered triple consisting of a set called the **domain**, a set called the **codomain**, and a called the **graph**. A graph is a (possibly empty) set consisting solely of ordered pairs of an element in the domain and an element in the codomain such that for each element x in the domain there is exactly one ordered pair containing x.

**Exercise 1.4: i. Show that if a function has empty domain (i,e. the domain is the empty set), then its graph is empty (i.e. the graph is the empty set).

ii. Show that a function cannot have empty codomain but nonempty domain.

Definition 1.5: A function is **injective** or **one-to-one** if for each element in the codomain, there is at most one ordered pair in the graph containing that element. A function is **surjective** or **onto** if for each element in the codomain, there exists an ordered pair in the graph containing that element. A function that is both injective and surjective is called **bijective**.

Definition 1.6: The **image** of a function is the set of elements in the codomain that are each contained in some tuple in the graph.

Exercise 1.7: A function is surjective iff (if and only if) its image and codomain are the same.

Definition 1.8: The **inverse image** of a set, A, in the codomain of f is the set of points x in the the domain such that $f(x) \in A$. We denote this set $f^{-1}(A)$.

Exercise 1.9: i. A function is surjective iff the inverse image of any nonempty set is nonempty.

ii. A function is injective iff the inverse image of any singleton (a set consisting of one element) is a singleton or empty.

Definition 1.10: The **union** of a family of sets, $\{X_i\}_{i \in I}$, which we write as $\bigcup_{i \in I} X_i$ (where I denotes an index set) is the set $\{x | \exists i \in I \, x \in X_i\}$. The **intersection** of a family of sets, $\{X_i\}_{i \in I}$, which we write as $\bigcap_{i \in I} X_i$, is the set $\{x | \forall i \in I \, x \in X_i\}$. For finite union (respectively intersection) we may write $X_1 \cup ... \cup X_2$.

Exercise 1.11: What is $\{1, 2, 3\} \cup (3, 4, 5\}$? What is $\{1, 2, 3\} \cap \{3, 4, 5\}$. What is $\{1, 2, 3\} \cup \{1, 2, 3\}$? What is $\{1, 2, 3\} \cap \{4, 5, 6\}$?

*Exercise 1.12: What is the union of an empty family of sets (i.e. $\{X_i\}_{i \in \emptyset}$)?

Definition 1.13: Two sets are **disjoint** if there intersection is empty. A is a **subset** of B if $\forall x \in A$, $x \in B$.

Exercise 1.14: Show that any set is a subset of interself.

Definition 1.16: The **empty set**, which we denote $\emptyset$, the set containing no elements.

Exercise 1.15: Show that the empty set is a subset of every other set.

Definition 2.1: A **topology** on a set X is a subset of $P(X)$ (i.e. the power set of X, which is the set of all subsets of X) that is closed under arbitrary union and finite intersection. We call element of $P(X)$ in the topology and **open sets**. We call a set X a **topological space,** or simply a space, if it endowed with a topology. More formally, an ordered pair with two elements, a set and a topology on that set, is a topological space. Note, that these two definitions actually specify different sets. Unless otherwise notes, if we are refer to a topological space, we are referring to the underlying set . The set of all open sets in a topology will be denoted $Op(X)$.

Proposition 2.2: For any topology, T, $X \in T$ and $\emptyset \in T$.

Proof: For the former take the empty intersection of sets in the topology and for the latter take the empty union of sets in the topology.

Note on notation: If an exercise is written as a statement (for example, “Any set with property A also has property B”), then the purpose of the exercise is true prove that statement)

Exercise 2.3: For any set X, there is a bijection between the set of all topologies on X and $P(X)$.

Definition 2.4: For a set X, the discrete topology is the topology consisting of all subsets of $P(X)$. The indiscrete topology is the topology consisting only of X and $\emptyset$.

Exercise 2.5: i. Show that the **discete topology** and the **indiscete topology** are actually toplogies.

ii. On what set are the discete topology and the indiscrete topology the same?

Definition 2.6: For two topologies T_1 and T_2on a set X, T_1 is **finer** than T_2 if $T_1 \supset T_2$. T_1 is **coarser** than T_2 is $T_1 \subset T_2$. T_1 and T_2 are comparable if one is finer than the other.

Exercise 2.7: i. For any topology, T, there is exactly one topology that is both finer and coarser than T.

ii. Give an example of two topologies (on the same set) that are not comparable.

Example 2.8: The discete topology is finest topology. That is, the discrete topology is finer than every other topology. The indiscete topology is the coarsest topology. To put this another way, in the set of topologies (on a set) ordered by the relation ($T > T'$ is T is finer than T), the discrete topology is the greatest element and the indiscrete topology is the least element.

Definition 3.1: A subset of X is **closed** (in a topology T) if it is the complement of an open set.

Proposition 3.2: The set of all closed sets is closed under the operations of finite union and arbitrary intersection.

The following proof relies on De Morgan's Laws, which state $X \setminus \bigcup_{i\in I} A_i = \bigcap_{i\in I} X \setminus A_i$ and $X \setminus \bigcap_{i\in I} A_i = \bigcup_{i\in I} X \setminus A_i$ (where $X \setminus Q$ denotes all of the elements of X that are not in Q and $Q \subset X$)

It is a good exercise to try to prove De Morgan's Laws on your own, but I will prove them on Saturday (hint: two sets are equal if and only if they contain the same elements, meaning they are subsets of each other).

Proof: We denote a closed set by C and an open set by U.

$\bigcup_{i=1}^{n} C_i = \bigcup X \setminus U_i = X \setminus \bigcap_{i=1}^{n} U_i = X \setminus U' = C'$, which is closed.

$\bigcap_{i\in I} C_i = \bigcap_{i\in I} X \setminus U_i = X \setminus \bigcup_{i\in I} U_i = X \setminus U' = C'$, which is closed.

Exercise 3.3: Any topology can be specified by a collection of subsets of $P(X)$ that is closed under arbitrary intersection and finite union.

Definition 3.4: A **basis** $\boldsymbol{B}$ for a topology T on X is a subset of T that satisfies the following properties:

(1) $\forall x \in X\, \exists B \in \boldsymbol{B}$ such that $x \in B$.

(2) For B_1, $B_2 \in \boldsymbol{B}$, and any point $x \in B_1 \cap B_2$ there is a B_3 is $\boldsymbol{B}$ such that $B_3 \subset B_1 \cap B_2$ and $x \in B_1 \cap B_2$.

(3) For all $U \in Op(x)$ and $\forall x \in U\ \exists B \in \boldsymbol{B}$ such that $B \subset U$ and $x \in B$.

*Exercise 3.5: Any subset of $P(X)$ satisfying conditions 1 and 2 specifies a topology (so show that such a set is closed under finite intersection and arbitrary union).

Lemma 3.6: For any topology with a basis, any open set U is the union of basis elements that are subset of U.

Lemma 3.7: A set is a basis for at most one topology.

Proposition 3.8: The folowing are equivalent (for a set of statements, we say statements are equivalent if any statements implies all of of the other statements):

(1) T is finer than T'.

(2) There exists bases $\boldsymbol{B}$ and $\boldsymbol{B'}$for T and T' (respectively) such that for any element B' of $\boldsymbol{B'}$and any $x \in B'$, we can find a $B \in \boldsymbol{B}$ such that $B \subset B'$ and $x \in B$.

Proof: $1 \implies 2$: Suppose 1 is true. Then every open set in T' is open in T. Since $\boldsymbol{B'} \subset T'$ and $T' \subset T$, $\boldsymbol{B'} \subset T$. We are done by condition 3 of definition 3.4.

$2 \implies 1$: Take an open set, U in T. Write this open set as a union of basis elements. For each element in the union there exists a $B' \in \boldsymbol{B'}$ containing that element, so by assumption we can find a $B \in \boldsymbol{B}$ containing that element. For each $x \in U$, choose such a B containing X. Thus, U is the union of elements of $\boldsymbol{B}$, meaning it is open in T.

Side note 3.9: We are using the axiom of choice in the second part of the proof (where?). If we wanted to avoid using the axiom of choice, we could reformulate the proof as a proof by contradiction (how?).

Definition 3.10: A **subbasis** or **subbase** is a subset, S, of $P(X)$ such that $\bigcup_{s\in S} s = X$.

*Exercise 3.11: For any subbasis there is a finest topology containing this subbasis. We say that this topology is the topology **generated** by the subbase.

Proof 1: Constructive proof

Proof 2: Show that the arbitrary intersection of a set of topologies is either empty or a topology.

Defition 4.1: (simplified) The **Cartesian product** of a family (synonym for set) of nonempty sets, $\{X_i\}_{i\in I}$ is the ordered collection of all sets containing exactly one element from each X_i and no other elements. We call an ordered set of 1 element from each x_i a **tuple**.

Definition 4.2: (abstract) The **Cartesian product** of a family (synonym for set) of sets, $\{X_i\}_{i \in I}$ is the collection of all functions with domain I and codomain $\bigcup_{i \in I} X_i$ such that $f(i) \in X_i$.

** Exercise 4.3: Under definition 4.2, what the cartesian product of the empty set (we call such a product, the empty product)? What is the Cartesian product of $\{X_i\}_{i \in I}$ if $\exists i \in I$ such that $X_i = \emptyset$. (To solve this problem, you must apply the definition of a function (1.3). Namely, how many (if there are any at all) functions with empty domain and empty codomain? What about function with nonempty domain but empty codomain?)

Definition 4.4: For a family of topological spaces, the **product topology** on the cartesian product of the underlying power sets is the topology generated by taking as a subbase all tuples such that each element in the tuple is an open set in the respective topological space and all but finitely many elements in the tuple are the entire topological space.

Definition 4.5: The **projection ma**p π_i from a Cartesian product is the map sending an element of the Cartesian product to its i^{th} coordinate. The product topology is the topology by taking as a subbase $\{\pi_i^{-1}(U) | \, i \in I \,\text{and}\, U \in Op(X_i)\}$.

* Exercise 4.6: Show that definitions 4.4 and 4.5 are equivalent (that is, show that the topologies generated in each definition are the same. To do this, one must be able to construct a topology from a subbase, so you should already have done a constructive proof of exercise 3.11).

Exercise 4.7: Show that the subbase in definition 4.4 is actually a basis.

Definition 5.1: For a subset, A, of a space X, the **subspace topology** is that generated (by the subbase) of all sets of the form ($U \cap A$ where $U \in Op(X)$).

Exercise 5.2: Show that every open set in the subspace topology can be written as $U \cap A$ for some $U \in Op(X)$.

Exercise 5.3: Every closed set in the subspace topology can be written as $C \cap A$ where C is a closed set (in the topology on X). For any closed set C in X, $C \cap A$ is closed in A.

Exercise 5.4: Show that if A is open, then the subspace topology is simply the family of open sets in X that are subsets of A.

Definition 6.1: The **interior** of an element B of $P(X)$ is the greatest (in respect to the subset ordering; $A < B$ if $A \subset B$) open set that is a subset of B.

Definition 6.2: The **closure** of an element B of $P(X)$ is the least closed set that is a superset of B (that is, the least closed set containing B).

Exercise 6.3: For any $B \in P(X)$, the interior is the union of all open sets contained in B and the closure is the intersection of all closed sets containing B. The converse is true for both statements.

Notation 6.4: We will denote the closure of B by $\overline{B}$ and the interior of B by $\text{Int}(B)$.

Example 6.5: For any space X, $\overline{X} = X$, $\text{Int}(X){=}X$, $\overline{\emptyset} = \emptyset$, $\text{Int}(\emptyset){=}\ \emptyset$.

Exercise 6.6: For any point x and set B, $x \in \overline{B}$ iff every open set containing x intersects B.

Definition 6.7: x is a limit point of B iff every open set containing x intersects B at a point other than x itself.

Exercise 6.8: $x \in \overline{B}$ iff $x \in B$ or x is a limit point of B.

Exercise 6.9: For any closed set, A, $\overline{A} = A$.

Lemma 6.10: A set is closed iff it contains all of its limit points.

Definition 7.1: A function between two topological spaces (i.e. the domain and codomain are each sets with a topology) is **continuous** if the inverse image of any open set is open.

Example 7.2: Any function (between topological spaces) with domain that is discrete is continuous.

Definition 7.3: A function is **open** if the image of any open set is open.

Example 7.4: Any function with codomain that is discrete is open.

Definition 7.5: A function is **closed** if the image of any closed set is closed.

Exercise 7.6: A bijective function is continuous iff the inverse function is open.

Exercise 7.7: A bijective function is open iff it is closed.

Definition 7.8: A **homeomorphism** is a bijection that is open and continuous.

Side note 7.9: In the category of topological spaces, the morphisms are continuous functions and a morphism in the category is a homeomorphism iff it is an isomorphism.

Proposition 7.10: The composition of two continuous functions is a continuous function.

Proof: Take continuous function: $f : G \to H$ and $g : H \to I$. We note that $(f \circ g)^{-1}(A) = f^{-1}(g^{-1}(A))$ (prove this!). Thus, for an open set U in I, $(f \circ g)^{-1}(U) = f^{-1}(g^{-1}(U))$, which is the inverse image under f of an open set, since g is continuous. Since f is continous, such a set is open.

Definition 7.11: A **cover** of a set X, is a set of open sets, U_i in X, such that $\bigcup U_i = X$ (more precisely this is an open cover). A **subcover** of a cover is a cover that is a subset of the original cover.

Proposition 7.11 (gluing): Suppose we have two spaces X and Y and a cover of X such that on each element of the cover we have a continuous function into Y defined on that open set, and such that for any two elements in the cover, the respective functions agree on their intersection. Then there exists a continuous function from X into Y such that the function restricted to any element of the cover agree with that element of the cover.

Proof: Define $f(x) = f_i(x)$ where f_i is a function on an element of the cover containing x. f is well-defined since if any two of our functions agree on their intersection. f is defined everywhere since we have a cover. For any open set U in Y, denoting elements in our cover by C_i $f^{-1}(U) = \bigcup_{i \in I}(f^{-1}(U) \cap C_i) = \bigcup_{i \in I}(f_i^{-1}(U) \cap C_i)$, which is open since each f_i is continuous and each C_i is open.

Now let us look at an actually example of a topological space!

Let us take the set of real numbers, $\boldsymbol{R}$, (we will discuss a formal construction of the reals) with the topology generated by **open balls**. That is, sets of the form $\{x \mid |x_0 - x| < \epsilon\}$, which we will denote $B(x, \epsilon)$.

Exxercise 8.1: What we just specified is actually a basis.

Exercise 8.2: Any open set that is not $\emptyset$ or X can be written as a disjoint union of open intervals, that is, sets of the for the $\{x | a < x < b\}$.

Lets give some properties and show that $\boldsymbol{R}$ satisfies those properties.

Definition 8.3: A space is **Hausdorff** or T_2 if for any two distinct points, there are disjoint open sets containing each.

Proposition 8.4: $\boldsymbol{R}$ is hausdorff.

Take two points, $a - b$. $|a - b| = \delta$, so take the open sets $B(a, \delta/2)$ and $B(b, \delta/2)$. (Check that these sets are actually disjoint.)

Definition 8.5: A space is said to be **compact** if every open cover has a finite subcover.

Proposition 8.5: $\boldsymbol{R}$ is not compact.

Take the open cover consisting of $B(z, 1)$ for $z \in \boldsymbol{Z}$.

Now we will prove the first "hard" result of this course (note that statements 2 and 3 are named after mathematicians!).

** Proposition 8.6: The following are equivalent for a subset A of R:

(1) A is compact.

(2) A is closed and bounded (Heine-Borel).

(3) Any sequence of points in A has a convergent subsequence (Bolzano-Weierstrass).

Proof: $1 \implies 2$: Exercise.

$2 \implies 3$: Claim 1: On any bounded set in R, for any sequence on that set, we can find a monotonic subsequence.

Since this subsequence is on a bounded set in A, it has a least upper bound (respectively greatest lower bound if our sequence if decreasing). Such a least upper bound, x, is a limit point of A, so since A is closed, $x \in A$. Our subsequence converges to x.

$3 \implies 1$: Exercise.