Sequences and Series

Arithmetic Series

An arithmetic progression is a sequence $a_1, a_2, \ldots, a_n$ such that the additive difference between two consecutive terms is a constant d (known as the common difference.)

The method for summing an arithmetic progression $a_1, a_2, \ldots, a_n$ comes from a simple concept:

$$\text{average} = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

But the average of an arithmetic progression is just $\frac{a_1+a_n}{2}$! Thus, we have a formula for the sum of an arithmetic progression:

$$a_1 + a_2 + \ldots + a_n = n\left(\frac{a_1 + a_n}{2}\right).$$

Problem 1 (Putnam 1979)**.** Let $x_1, x_2, \ldots$ be a sequence of nonzero real numbers satisfying $x_1 = a, x_2 = b$, and

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}$$

for $n \geq 3$. For what values of a, b is x_n an integer for infinitely many values of n?

Geometric Series

A geometric progression is a sequence $a_1, a_2, \ldots, a_n$ such that the quotient of any consecutive terms $\frac{a_{k+1}}{a_k}$ is a constant r (known as the common ratio). Here we will find an expression for the sum of a geometric sequence:

Let $S = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1}$. Then $rS = a_1 r + a_1 r^2 + \ldots + a_1 r^n$.
So $S - rs = a_1 - a_1 r^n$.
So $S = \frac{a_1 - a_1 r^n}{1-r}$.

If the geometric series has an infinite number of terms, then it has a finite value iff $-1 < r < 1$. This is because in the expression

$$S = \frac{a_1 - a_1 r^n}{1 - r},$$

the $a_1 r^n$ term must take on a finite value. If $|r| > 1$ then $a_1 r^n$ has infinite magnitude; if $r = 1$ the denominator is zero and so the sum is infinite; and if $r = -1$ then S alternates between 0 and a_1, i.e. it has no stable value. We conclude that $|r| < 1$.

Problem 2. Find an infinite geometric series whose sum is 6 and such that each term is four times the sum of all the terms that follow it.

Problem 3. Two infinite geometric series $a + aq + aq^2 + \ldots$ and $b + br + br^2 + ...$ have the same sum. Also, the second term of the second series is the first term of the first series. If $r = \frac{1}{2}$ for the second series, find r for the first series.

Problem 4. Evaluate

$$\sum_{i=1}^{\infty} \frac{i}{5^i}.$$

Base problems are often geometric series in disguise.

Problem 5. For each integer $n \geq 4$, let a_n denote the base-n number $0.\overline{133}_n$. The product $a_4 a_5 \ldots a_{99}$ can be expressed as $\frac{m}{n!}$, where m and n are positive integers and n is as small as possible. What is the value of m?

(A)98 (B)101 (C)132 (D)798 (E)962

Recursion

Sequences are often defined **recursively**, i.e. by supplying a formula for the term x_n in terms of earlier terms. Often in math contests we are asked to find a **closed form** for the term x_n. Alternatively, we might be asked to find an expression for the sum of terms or to evaluate a certain sum.

Perhaps the most famous recursion is the **Fibonacci sequence**, which is a sequence of integers defined as $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$.

Problem 6. If F_n is the Fibonacci sequence, i.e. the sequence with $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$, then evaluate

$$\sum_{n=1}^{\infty} \frac{F_n}{3^n}.$$

Telescopic Series

When we are asked to evaluate an ugly sum, a common strategy is to try to manipulate the term being summed to make it **telescope**, i.e. to make parts of it cancel out between one term and the next. Ideally, we can manipulate the sum into the form

$$\sum_{k=1}^{n} [F(k+1) - F(k)]$$

since this evaluates to $F(n+1) - F(1)$.

Problem 7. Given that $a_1, a_2, \ldots, a_n$ are positive numbers in arithmetic progression, show that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_{n-1}} a_n = \frac{n-1}{a_1 a_n}.$$

Problem 8. Given that $x_1, x_2, \ldots$ is a sequence satisfying the conditions $x_1 = \frac{1}{2}$ and $x_{k+1} = x_k^2 + x_k$, find the greatest integer less than

$$\frac{1}{x_1 + 1} + \frac{1}{x_2 + 1} + \ldots + \frac{1}{x_{100} + 1}.$$

Defining Secondary Sequences

Another strategy for dealing with recursion is to define a secondary sequence that has simple properties, just as we defined secondary polynomials two weeks ago when dealing with complicated polynomials. Here are a few examples:

Problem 9 (AIME 2005II #11)**.** Let m be a positive integer, and let $a_0, a_1, \ldots, a_m$ be a sequence of integers such that $a_0 = 37, a_1 = 72, a_m = 0$, and $a_{k+1} = a_{k-1} - \frac{3}{a_k}$ for $k = 1, 2, \ldots, m-1$. Find m.

Problem 10 (AIME 2007I #14)**.** Let a sequence be defined as follows: $a_1 = 3, a_2 = 3$, and for $n \geq 2, a_{n+1}a_{n-1} = a_n^2 + 2007$. Find the largest integer less than or equal to

$$\frac{a_{2007}^2 + a_{2006}^2}{a_{2007} a_{2006}}.$$

Although there is no guaranteed way to find such secondary sequences, here are a few strategies to try:

- Eliminate fractions by multiplying by the denominator or by substituting $y_n = \frac{1}{x_n}$.
- Try writing several 'copies' of the recursive equation for $n = k, k+1, k-1$, etc. See if you can use algebraic manipulation to get a new recursive expression that is simpler.

- In particular, if there is a constant in the recursion that looks arbitrary (like 2007 in the above problem), it probably is! Try eliminate that arbitrary constant, using the previous bullet point.

Ideally, we want to find an **invariant**, i.e. an expression that is independent of n. This is good because then we can calculate the invariant using small values of n.

Sequences and series arise in a variety of applications, among them combinatorics. As a final example, consider this problem which requires us to find a recursion and solve it by examining a secondary sequence that, it turns out, is geometric:

Problem 11. An ant is at one vertex of an equilateral triangle. Every second, the ant randomly picks one of the two other vertices of the triangle and moves to that vertex. What is the probability that after ten seconds, the ant is at its starting point?

Problems

Problem 12. If the integer k is added to each of the numbers 36, 300, and 596, one obtains the squares of three consecutive terms of an arithmetic series. Find k.

Problem 13 (AIME)**.** A sequence of numbers $x_1, x_2, x_3, \ldots, x_{100}$ has the property that, for every integer k between 1 and 100, inclusive, the number x_k is k less than the sum of the other 99 numbers. Given that $x_{50} = m/n$, where m and n are relatively prime positive integers, find $m + n$.

Problem 14 (AMC12 2000 #16)**.** A checkerboard of 13 rows and 17 columns has a number written in each square, beginning in the upper left corner, so that the first row is numbered 1, 2, ..., 17, the second row 18, 19, ..., 34, and so on. If the board is renumbered down the columns (so the left column is 1, 2, ..., 13, and so on), find the sum of the values in the squares which do not change numbers.

Problem 15 (AMC 12 2007B #15)**.** The geometric series $a + ar + ar^2 + \ldots$ has a sum of 7, and the terms involving odd powers of r have a sum of 3. What is $a + r$?

$$\text{(A) } \frac{4}{3} \qquad \text{(B) } \frac{12}{7} \qquad \text{(C) } \frac{3}{2} \qquad \text{(D) } \frac{7}{3} \qquad \text{(E) } \frac{5}{2}$$

Problem 16 (HMMT 2005 Guts #3)**.** Find the sum

$$\frac{2^1}{4^1 - 1} + \frac{2^2}{4^2 - 1} + \frac{2^4}{4^4 - 1} + \frac{2^8}{4^8 - 1} + \cdots.$$

Problem 17 (AHSME 1999 #20)**.** The sequence $a_1, a_2, \ldots, a_n$ satisfies $a_1 = 19$, $a_9 = 99$, and, for all $n \geq 3$, a_n is the arithmetic mean of the first $n - 1$ terms. Find a_2.

Problem 18 (AIME 2007II #12)**.** The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that

$$\sum_{n=0}^{7} \log_3(x_n) = 308 \quad \text{and} \quad 56 \leq \log_3\left(\sum_{n=0}^{7} x_n\right) \leq 57,$$

find $\log_3(x_{14})$.

Problem 19 (AIME)**.** Suppose n is a positive integer and d is a single digit in base 10. Find n if

$$\frac{n}{810} = 0.d25d25d25\ldots$$

Problem 20 (AIME 2000I #10)**.** A sequence of numbers $x_1, x_2, x_3, \ldots, x_{100}$ has the property that, for every integer k between 1 and 100, inclusive, the number x_k is k less than the sum of the other 99 numbers. Find x_{50}.

Problem 21 (AIME)**.** An infinite geometric series has sum 2005. A new series obtained by squaring each term of the original series has 10 times the sum of the original series. Find the common ratio of the the original series.

Problem 22. An ant is at one vertex of an regular tetrahedron. Every second, the ant randomly picks one of the two other vertices of the tetrahedron and moves to that vertex. What is the probability that after ten seconds, the ant is at its starting point?

Problem 23 (Mathematical Olympiad Treasures)**.** The sequence a_n is defined as follows: $a_1 = a_2 = 1, a_3 = 199$, and

$$a_{n+1} = \frac{1989 + a_n a_{n-1}}{a_{n-2}}.$$

Prove that a_n is an integer for all positive integers n.

Problem 24 (Telescoping works for products too!)**.** Prove that

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{n^2}\right)=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots=\frac{1}{2}.$$

Problem 25 (AIME 1993, #11)**.** Alfred and Bonnie play a game in which they take turns tossing a fair coin. The winner of a game is the first person to obtain a head. Alfred and Bonnie play this game several times with the stipulation that the loser of a game goes first in the next game. Suppose that Alfred goes first in the first game, and that the probability that he wins the sixth game is m/n, where m and n are relatively prime positive integers. What are the last three digits of $m+n$?

Problem 26 (Mathematical Olympiad Challenges)**.** Let F_n be the Fibonacci sequence ($F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1}$). Evaluate

$$\sum_{n=2}^{\infty}\frac{F_n}{F_{n-1}F_{n+1}} \qquad \text{and} \qquad \sum_{n=2}^{\infty}\frac{1}{F_{n-1}F_{n+1}}.$$

Challenge Problems

Problem 27. A sequence is defined by $a_1 = 1995, a_{n+1} = \frac{a_n}{a_n+1}$ for all $n \geq 1$. If $a_{1995} = \frac{m}{n}$, where m and n are relatively prime integers, find the remainder when $m+n$ is divided by 1000.

Problem 28 (Art and Craft of Problem Solving)**.** Find a formula for the sum $r + 2r^2 + 3r^3 + ... + nr^n$, and generalize.

Problem 29 (HMMT Guts 2006)**.** Beginning at a vertex, an ant crawls between the vertices of a regular octahedron. After reaching a vertex, it randomly picks a neighboring vertex (sharing an edge) and walks to that vertex along the adjoining edge (with all possibilities being equally likely). What is the probability that after walking along 2006 edges, the ant returns to the vertex where it began?

Problem 30 (1969 AHSME)**.** Let S_n and T_n be the respective sums of the first n terms of the two arithmetic series. If

$$\frac{S_n}{T_n}=\frac{7n+1}{4n+27}$$

for all n, the ratio of the eleventh term of the first series to the eleventh term of the second series is:

(A) 4 : 3 (B) 3 : 2 (C) 7 : 4 (D) 78 : 71 (E) undetermined

Problem 31 (HMMT 2003 Algebra #8)**.** Find the value of $\frac{1}{3^2+1}+\frac{1}{4^2+2}+\frac{1}{5^2+3}+\ldots$. (Hint: Try to telescope this sum.)

Problem 32 (Putnam)**.** Prove that

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\frac{2}{3}.$$

Problem 33 (HMMT 2006 Algebra #9)**.** Compute the value of the infinite series

$$\sum_{n=2}^{\infty}\frac{n^4+3n^2+10n+10}{2^n\cdot(n^4+4)}.$$

(Hint: try to factor the n^4+4 term in the denominator, then look for an expansion that will telescope.)