Roots and Coefficients

Roots and Remainders

A polynomial of degree n takes the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

If r is a value of x that makes $f(x) = 0$, then r is said to be a **root** or a **zero** of the polynomial. Then we can factor $(x - r)$ out of the polynomial, leaving a quotient polynomial and no remainder:

$$f(x) = q(x) \cdot (x - r).$$

If c is not a root of $f(x)$, then division by $(x - c)$ leaves a remainder:

$$f(x) = q(x) \cdot (x - c) + r(x).$$

Problem 1 (USSR Olympiad)**.** Find the remainders upon dividing $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$

a. by $x - 1$.

b. by $x^2 - 1$.

Problem 2 (AHSME 1976 #19)**.** A polynomial $p(x)$ leaves remainder 3 when divided by $x-1$ and remainder 5 when divided by $x - 3$. What is the remainder when it is divided by $(x - 1)(x - 3)$?

Division by $(x-r)$ where r is a root leaves a quotient with degree $n-1$. Repeating this process and factoring further terms of the form $(x - r_i)$ out of $q(x)$ until there are no roots left (i.e. until we are left only with a constant), we have:

$$f(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n). \tag{1}$$

(Note that some of the roots may be repeated, and they will not necessarily all be real. But for a degree-n polynomial, there will be n of them!)

Factoring

An important part of solving many polynomial problems is ascertaining the roots, i.e. factoring out terms. Here are some useful factorizations to remember:

- $ab + a + b + 1 = (a + 1)(b + 1)$
- $x^2 - y^2 = (x + y)(x - y)$
- $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
- $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
- $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1})$
- For odd n only: $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \ldots - xy^{n-2} + y^{n-1})$. (Why doesn't this work for evens?)

Problem 3. Find all real pairs (x, y) in this system:

$$\begin{cases} x^2y + xy^2 &= 30 \\ x^3 + y^3 &= 35 \end{cases}$$

Problem 4. Given that x and y are real numbers satisfying $\frac{1}{x} + \frac{1}{y} = -\frac{1}{6}$, and that

$$x^2 + 3x + y^2 + 3y + 2xy = 4,$$

find all possible ordered pairs (x, y).

Viete's Formulas

We stated earlier that

$$f(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n).$$

Expanding the right-hand side yields

$$\begin{aligned} f(x) &= a_nx^n - a_n(r_1 + r_2 + \ldots + r_n)x^{n-1} + a_n(r_1r_2 + r_1r_3 + \ldots + r_{n-1}r_n)x^{n-2} - \ldots + (-1)^n a_n r_1r_2 \cdots r_n. \\ &= a_nx^n - a_n\left(\sum r_i\right)x^{n-1} + a_n\left(\sum r_ir_j\right)x^{n-2} - \ldots + (-1)^n a_n\left(r_1r_2 \cdots r_n\right). \end{aligned}$$

Comparing term by term with $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, we have the following relations:

$$\begin{aligned} a_{n-1} &= -a_n(r_1 + r_2 + \ldots + r_n) \\ a_{n-2} &= a_n(r_1r_2 + r_1r_3 + \ldots + r_{n-1}r_n) \\ &\vdots \\ a_1 &= (-1)^{n-1}a_n(r_1r_2 \cdots r_{n-1} + r_1r_2 \cdots r_{n-2}r_n + \ldots + r_2r_3 \cdots r_n) \\ a_0 &= (-1)^n a_n(r_1r_2 \cdots r_n) \end{aligned}$$

Rearranging, we obtain formulas for sum of the product of the zeroes taken k at a time, $1 \leq k \leq n$. These equations are known as Viete's formulas:

$$\begin{aligned} r_1 + r_2 + \ldots + r_n &= -\frac{a_{n-1}}{a_n} \\ r_1r_2 + r_1r_3 + \ldots + r_{n-1}r_n &= \frac{a_{n-2}}{a_n} \\ &\vdots \\ r_1r_2 \cdots r_{n-1} + r_1r_2 \cdots r_{n-2}r_n + \ldots + r_2r_3 \cdots r_n &= (-1)^n \frac{a_0}{a_n} \end{aligned}$$

Problem 5 (AMC12 2001 #19)**.** The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. If the y-intercept of the graph of $y = P(x)$ is 2, what is b?
(A) -11 (B) -10 (C) -9 (D) 1 (E) 5

Newton's Sums

For $1 \le k \le n$, let r_i be the roots of $f(x) = \sum_{i=1}^{n} a_i x^i$ and let

$$S_k = r_1^k + r_2^k + \cdots + r_n^k.$$

Then the following relations apply:

$$\begin{aligned} a_n S_1 + a_{n-1} &= 0 \\ a_n S_2 + a_{n-1} S_1 + 2a_{n-2} &= 0 \\ a_n S_3 + a_{n-1} S_2 + a_{n-2} S_1 + 3a_{n-3} &= 0 \\ &\vdots \end{aligned}$$

The first equation is a rearrangement of Viete's formula, and the others follow by induction. The details are left as one of the exercises for today.

Symmetric Polynomials

Let us reframe the question of calculating sums like

$$r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4$$

by considering these as **symmetric polynomials** in $r_1, r_2, r_3, \ldots$.

A polynomial $f(x, y)$ is symmetric if $f(x, y) = f(y, x)$. For more than two variables, the requirement is that switching any two variables (and so, by extension, any or all of the variables for one another) yields the same polynomial. Here are some examples of important symmetric polynomials:

- The elementary symmetric polynomials $\sigma_1 = x + y + z, \sigma_2 = xy + yz + zx, \sigma_3 = xyz$.
- The Newton sums $S_2 = x^2 + y^2 + z^2$, etc.

Symmetric polynomials can be expressed in terms of the elementary symmetric polynomials. So a general strategy is to:

a. Rewrite a symmetric polynomial expression $f(r_1, r_2, \ldots, r_n)$ in terms of elementary symmetric polynomials $\sigma_1, \sigma_2, \ldots, \sigma_n$.

b. Use the rewritten expressions to calculate $\sigma_1, \sigma_2, \ldots, \sigma_n$.

c. Find the roots of the polynomial $x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} - \ldots + (-1)^n \sigma_n$. These are $r_1, r_2, \ldots, r_n$.

Problem 6. Factor $x^3 + y^3 + z^3 - 3xyz$ into elementary symmetric functions.

Problem 7. Solve the system $x^4 + y^4 = 82, x + y = 4$.

Substitution

Semi-obvious point: the sum of the coefficients of $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is $f(1)$. In general, it helps to try plugging in special values like 0, 1, -1, i, etc.

Problem 8 (Brainteaser)**.** Alice and Bob are playing a game. Alice thinks of a secret polynomial $p(x)$ with non-negative integer coefficients. Bob chooses an integer a and Alice tells Bob the value $p(a)$. Then Bob chooses an integer b and Alice tells Bob the value $p(b)$. Now Bob can tell Alice the polynomial $p(x)$ correctly. What is Bob's strategy for determining the polynomial $p(x)$?

Operation on Roots

Sometimes we are given a polynomial and asked to modify it so its roots are

Problem 9. Given that $5x^4 - 2x^3 - 21x^2 + 8$ has roots a, b, c, d, find a polynomial with roots $-\frac{1}{a}, -\frac{1}{b}, -\frac{1}{c}$, and $-\frac{1}{d}$.

Problem 10. Suppose $P(x)$ is a polynomial of degree 8 with real coefficients and $P(k) = \frac{1}{k}$ for $k = 1, 2, \ldots, 9$. Determine the value of $P(10)$.

Problem 11 (2005 AIME2 #13)**.** Let $P(x)$ be a polynomial with integer coefficients that satisfies $P(17) = 10$ and $P(24) = 17$. Given that $P(n) = n + 3$ has two distinct integer solutions n_1 and n_2, find the product $n_1 \cdot n_2$.

Other Useful Theorems

The Rational Root Theorem. If $\frac{p}{q}$ is a rational root of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then p divides a_0 and q divides a_n.

Problem 12 (Engel)**.** The polynomial $ax^3 + bx^2 + cx + d$ has integral coefficients a, b, c, d, with ad odd and bc even. Show that at least one zero of the polynomial is irrational.

Descartes' Rule of Signs. If $f(x)$ is a polynomial with real coefficients, then

- the number of positive zeroes of $f(x)$ is equal to the number of variations in sign of $f(x)$, or less than this by an even number.
- the number of negative zeroes of $f(x)$ is equal to the number of variations in sign of $f(-x)$, or less than this by an even number.

A "variation in sign" in the polynomial is an instance where, if the polynomial's terms are ordered in decreasing (or increasing) order of power of x, neighboring coefficients have opposite signs. For instance, $f(x) = x^3 - 2x^2 - 2x + 1$ has two variations in sign.

In-Class Problems

Problem 13 (AHSME 1978 #13)**.** If a, b, c, d are nonzero numbers such that c and d are the roots of $x^2 + ax + b = 0$ and a and b are the roots of $x^2 + cx + d = 0$, find $a + b + c + d$.

Problem 14. A student awoke at the end of the algebra class just in time to hear the teacher say, "...and as a hint, the solutions to this polynomial form an arithmetic progression." Looking at the board, the student found a fifth-degree equation to be solved for homework, but he only had time to write down half the polynomial before it was erased:

$$x^5 - 5x^4 - 35x^3 + \dots$$

However, the student was still able to find the roots! What are they?

Problem 15 (AIME 1991 #8)**.** For how many real numbers a does the quadratic equation $x^2 + ax + 6a = 0$ have only integer roots for x?

Problem 16. Suppose $P(x)$ is a polynomial of degree $n \geq 1$ such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, \dots, n$. Find $P(n+1)$.

Problem 17 (AHSME 1973 #27)**.** If p, q, r are distinct roots of $x^3 - x^2 + x - 2 = 0$, then $p^3 + q^3 + r^3$ equals
(A) -1 (B) 1 (C) 3 (D) 5 (E) none of these

Problem 18. If the roots of a cubic polynomial $f(x) = x^3 - 4x^2 + 10x - 2$ are a, b, c, find

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}.$$

Problem 19 (AHSME 1981 #30)**.** If a, b, c, d are the solutions of the equation $x^4 + 15x^2 - 5x - 3 = 0$, then find an equation whose solutions are

$$\frac{a+b+c}{d^2}, \quad \frac{a+b+d}{c^2}, \quad ,\frac{a+c+d}{b^2}, \quad ,\frac{b+c+d}{a^2}.$$

Problem 20. Find the real solutions to the equation

$$\sqrt[5]{x} + \sqrt[5]{275 - x} = 5.$$

Problem 21 (Engel)**.** Let $f(x)$ be a polynomial with integral coefficients, and suppose that there exist distinct a, b, c, d such that $f(a) = f(b) = f(c) = f(d) = 5$. Prove that there is no integer k such that $f(k) = 8$.

Problem 22. Suppose $P(x)$ is a polynomial with integer coefficients and $P(a) = P(b) = P(c) = P(d) = 2$ for distinct integers a, b, c, d. Does there exist an integer k such that $P(k) = 1, 3, 5, 7$, or 9?

Problem 23 (AHSME 1976 #30)**.** How many distinct ordered triples (x, y, z) satisfy the equations

$$\begin{aligned} x + 2y + 4z &= 12 \\ xy + 4yz + 2xz &= 22 \\ xyz &= 6 \end{aligned}$$

Problem 24 (USAMO 1984)**.** The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.

Problem 25 (AIME 1990 #15)**.** Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

Problem 26 (Newton's Sums)**.** Prove the formulas for Newton's sums, which are, for $1 \le k \le n$,

$$a_n S_k + a_{n-1} S_{k-1} + \ldots + a_{n-k+1} S_1 + k a_{n-k} = 0.$$

Problem 27 (Polish Math Olympiad)**.** Find $x^7 + \frac{1}{x^7}$ given that $x + \frac{1}{x} = a$.

Problem 28 (USAMO 1974)**.** Let a, b, c be distinct integers, and let P be a polynomial with integer coefficients. Show that in this case the conditions

$$P(a) = b, \qquad P(b) = c \qquad P(c) = a$$

cannot be satisfied simultaneously.

Problem 29 (Canada 1971)**.** Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients. If $p(0)$ and $p(1)$ are both odd, prove that $p(x)$ has no integral roots.

Problem 30 (HMMT 2004 Algebra)**.** Find all real solutions to $x^4 + (2 - x)^4 = 34$.

Problem 31 (HMMT 2003 Algebra)**.** Suppose $P(x)$ is a polynomial such that $P(1) = 1$ and

$$\frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}$$

for all real x for which both sides are defined. Find $P(-1)$.

Problem 32 (AMC 12 2005 A #24)**.** Let $P(x) = (x-1)(x-2)(x-3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that

$$P(Q(x)) = P(x) \cdot R(x)?$$

(A) 19 (B) 22 (C) 24 (D) 27 (E) 32

Problem 33 (HMMT 2007 Algebra #6)**.** Consider the polynomial $P(x) = x^3 + x^2 - x + 2$. Determine all real numbers r such that there exists a complex number z not in the reals such that $P(z) = r$.

Problem 34. Let $f(x)$ be a polynomial of degree n, $n > 1$, with integer coefficients and n real roots, not all equal, in the interval (0,1). Prove that if a is the leading coefficient of $f(x)$, then $|a| \ge 2^n + 1$.

Problem 35 (HMMT 2007 #10)**.** The polynomial $f(x) = x^{2007} + 17x^{2006} + 1$ has distinct zeroes $r_1, \ldots, r_{2007}$. A polynomial P of degree 2007 has the property that $P\left(r_j + \frac{1}{r_j}\right) = 0$ for $j = 1, \ldots, 2007$. Determine the value of $P(1)/P(-1)$.