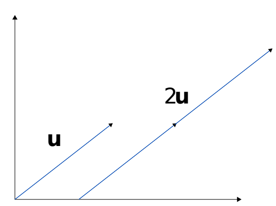
Fun with Vectors: Week 2 Notes

March 9, 2013

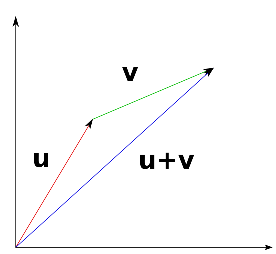
Vector Operations

- Scaling vectors: To scale vectors, draw a vector in the same direction with a different length.

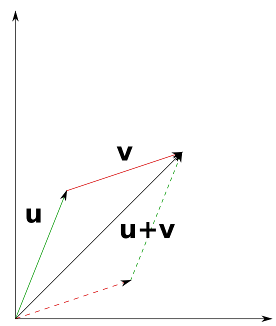


- Adding and subtracting vectors: To do this geometrically, recall that you can shift vectors around without changing the vector.

  You can add vectors by placing them head to tail

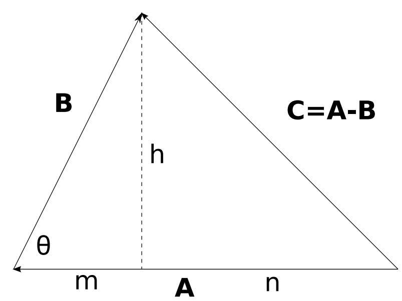


  or by using the parallelogram rule.



You should note that $\vec{u}+\vec{v}$ is called the **resultant vector**.

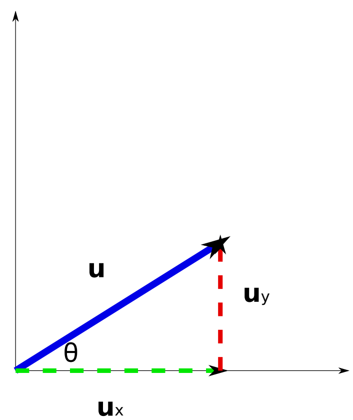
- Law of Cosines: When adding vectors geometrically, the law of cosines is often helpful. You can derive this using the Pythagorean theorem.



$$\begin{aligned} B^2 &= m^2 + h^2 \\ &= m^2 + C^2 - n^2 \\ &= m^2 + C^2 - (A^2 - 2mA + m^2) \\ &= C^2 - A^2 + 2mA \\ &= C^2 - A^2 + 2A\cos\theta \\ \implies & C^2 = A^2 + B^2 - 2A\cos\theta \end{aligned}$$

$$C^2 = h^2 + n^2 \qquad A = m + n \qquad m = B\cos\theta$$

- Vector Components: Any two-dimensional vector can be written as the sum of two **orthogonal**, or perpendendicular vectors. In order to do this, let's define two **unit vectors**, vectors which have a length of 1. The first is $\hat{\imath}$ which is a vector along the x-axis with a length of 1. The second is $\hat{\jmath}$ which is a vector along the y-axis.



From this picture, we can see that $\vec{u} = u_x\hat{i} + u_y\hat{j}$. Further, we can come up with a method for breaking vectors into components. Using trigonometry, we know that:

$$u_x = u\cos\theta$$
$$u_y = u\sin\theta$$

Components are so useful that we often use a different notation for them to make things more concise. We write

$$\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\vec{u} = u_x\hat{i} + u_y\hat{j} = u_x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

This is a very powerful technique, because it means that instead of adding vectors geometrically using the law of cosines, we can instead simply add and subtract their components. For example,

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

To scale vectors, we simply multiply each component by the scaling factor. For example,

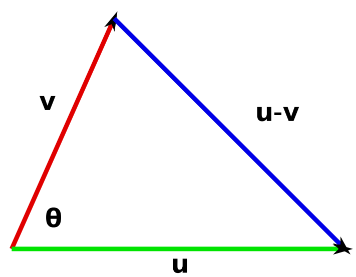
$$4\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \end{pmatrix}$$

- Inner Products: The **inner product**, or dot product, is a means of multiplying vectors to get a **scalar**, or number. This is useful in defining length. As a side note, we typically write the length of a vector $\vec{u}$ as $\|\vec{u}\|$.

  We are going to talk about the dot product in a slightly different way than what you're used to. We're going to specify several properties that we want the dot product to have. Then we're going to use these properties to figure out ways to calculate the dot product. Here are the properties we want the dot product to have:

  1. $\vec{u} \cdot \vec{u} = \|\vec{u}\|$, the dot product of a vector with itself is the length of the vector.
  2. $\vec{v} \cdot \vec{u} = \vec{u} \cdot \vec{v}$, the dot product is commutative (we'll see some examples of things that aren't commutative in a couple of weeks).
  3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u}\vec{v} + \vec{u}\vec{w}$, the dot product is distributive.

     Now we will figure out two ways to calculate the dot product. The first way is geometric. Using the properties of the dot product

     from above, we know that:

     $$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u}-2\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{v} = \|\vec{u}\|^2+\|\vec{v}\|^2-2\vec{u}\cdot\vec{v}$$

     Using the law of cosines (derived earlier), we know that:

     $$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

     By comparing these two equations, we can see that $\vec{u}\cdot\vec{v} = \|u\|\|v\| \cos\theta$.

     So now we have one way of calculating the dot product of two vectors. However, in order to use this formula, we have to know the direction and magnitude again! This is kind of annoying, because we just spent time trying to get rid of needing magnitude and direction by using components.

But all is not lost! We can figure out how to calculate the dot product using components pretty easily by figuring out how the dot product acts on the unit vectors $\hat{\imath}$ and $\hat{\jmath}$. Let's think about the properties of these unit vectors. The lengths of both are 1, and the directions are along the axes. The angle between $\hat{\imath}$ and $\hat{\imath}$ or $\hat{\jmath}$ and $\hat{\jmath}$ is 0, and the angle between $\hat{\imath}$ and $\hat{\jmath}$ is 90. Recall that $\cos 90 = 0$ and $\cos 0 = 1$ from the work we did last week with the unit circle (remember that cos is the x-coordinate).

So, using the geometric formula for the dot product, we know that $\hat{\imath} \cdot \hat{\imath} = 1$, $\hat{\jmath} \cdot \hat{\jmath} = 1$, and $\hat{\imath} \cdot \hat{\jmath} = 0$.

Keeping this in mind, let's consider the dot product of two vectors $\vec{u}$ and $\vec{v}$:

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= (u_x\hat{\imath} + u_y\hat{\jmath}) \cdot (v_x\hat{\imath} + v_y\hat{\jmath}) \\
&= u_x v_x \hat{\imath} \cdot \hat{\imath} + u_x v_y \hat{\imath} \cdot \hat{\jmath} + u_y v_x \hat{\jmath} \cdot \hat{\imath} + u_y v_y \hat{\jmath} \cdot \hat{\jmath} \\
&= u_x v_x + u_y v_y
\end{aligned}$$

So...the dot product can be calculated as the sum of the product of corresponding components! For example,

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \end{pmatrix} = (3)(4) + (2)(5) = 22$$