A Construction of the Numbers: The Naturals

Mandar Juvekar and Arian Nadjimzadah

1 Introductions

- Who: We are rising juniors at the University of Rochester. Both of us are majoring in mathematics and computer science.
- Contacting us: You can contact us via the teacher email for this class, `M14085-teachers@esp.mit.edu`. We do not bite (even virtually)! Please do not hesitate to email us with questions, comments, or even just to say hello.

2 How will this course run?

- Class sessions every Saturday at noon EDT.
- Notes containing material covered in class (and more) will be sent out after class.
- Class material will often build on previous weeks. Please do not hesitate to email us if you feel like you need to catch up.
- There is no homework! We might mention some things to think about if you are interested. Again, feel free to email if you would like to discuss those in greater detail.

3 What is a number?

- A way to count things.
- Measuring the physical world (length, weight, etc.).
- A means to making predictions.
- The basis for computation.

3.1 Some History

- Circa 35,000 BCE: Tally marks. Used to count simple things such as sheep going out to graze.
- 3,000 BCE: The Egyptians develop one of the first base-10 number systems.
- 1,750 BCE: The Babylonians develop a base 60 system (although with exceptions for certain numbers). This did not survive too long as it requires at least 60 different symbols and is difficult to do arithmetic with.
- 500 – 300 BCE: The Greeks and Indians (among other cultures) develop a basis for the study of whole number ratios.
  - Euclid's *Elements* published circa 300 BCE is one of the best known treatises that works with ratios. It is also the foundation of Euclidean geometry and formal reasoning.

- Some mentions of the existence of irrational numbers were present, but not widely accepted.[1]

- 300 – 200 BCE: The digits now used internationally make their appearance gradually in ancient India. The first formal use of the numeral 0 was seen in the first century CE.

If you are interested, [4] (more specifically section 4 in the article) has a nice overview of the development of the theory of numbers, and [1] contains an overview of the history of numbers from a more practical perspective.

4 The Need for Formalism

Let us think about defining a set. What is a set? Intuitively, a set is a collection of "objects". Is a set itself an "object"? Why not? Then a set of sets is also a set. Consider the following set:

$$\mathscr{R} = \text{The set of all sets which do not contain themselves}$$

According to our definition, $\mathscr{R}$ is a valid set. Does $\mathscr{R}$ contain itself? If it did, then $\mathscr{R}$ wouldn't be the set of all sets not containing themselves. Otherwise if $\mathscr{R}$ didn't contain itself then the definition of $\mathscr{R}$ dictates that $\mathscr{R}$ must contain itself. The set $\mathscr{R}$ is thus a paradox under our definition of "set".[2]

This demonstrates that intuitive definitions can sometimes lead to strange paradoxes. To avoid these, mathematicians go to great lengths to provide robust formal definitions for mathematical concepts. In the case of sets, these formal definitions led to Zermelo-Frenkel set theory, an axiomatic system to deal with sets. In the case of numbers, this led to the constructions we will be talking about.

Zermelo-Frenkel set theory is outside the scope of this course. However, it is still very cool, and if you are interested we recommend checking out [3], which gives a brief overview of the Zermelo-Frenkel axioms with some applications. The document also gives a set theoretic formalization of the natural numbers, which is an alternative formalization to the one we will look at.

4.1 Axiomatic Systems

All of your favorite theorems—perhaps the fundamental theorem of calculus or the Pythagorean theorem—are true because they can be proved. If you think about it, other true statements had to be used in the course of each of these proofs. Why can we trust *those* results? Because they were *also* proved. If we continue this process further and further back, it is clear that there must be some set of initial "truths" forming the foundation. These truths are known as **axioms**: statements that are assumed to be true without proof.

An axiomatic system is a neat way of formalizing many mathematical constructs. In an axiomatic system, we begin with a finite set of axioms which we consider to be true.[3] Other statements are said to be true if they can be logically derived from the axioms. True statements are often labelled **theorems**. Elementary ("Euclidean") geometry is an example of an axiomatic system, and so is the formalization of natural numbers we will see today.

4.2 A note on proofs from axioms

While we have defined axiomatic systems quite thoroughly, there still remains the question of how one goes about deriving new statements (theorems) from the axioms. Intuitively, we assume that any "logically made" mathematical argument that builds on the axioms and reaches the theorem we want to prove constitutes a valid proof. This isn't fully formal since "logical" is still not well defined. For the purposes of this class, we will work with the intuitive definition of "logical" that we all have from school. However for completeness (and because somebody asked us after class) we are writing this quick note.

[1] Pythagoras was a firm believer in the "absoluteness" of numbers, and believed that all physically relevant quantities come in whole number ratios. He even went so far as to commit murder to defend his beliefs!

[2] This paradox is known as Russel's paradox, and was discovered by Bertrand Russel at the turn of the twentieth century.

[3] Usually these are statements that one would think are "obviously correct". However it is possible (and interesting) to build and play with axiomatic systems with axioms that don't follow real-world intuition. So-called "non-Euclidean geometries" are examples of such systems.

In logic and set theory (which is the field of math that deals with this), axiomatic systems are almost always accompanied with a finite set of **rules of inference**. Rules of inference are symbolic rules that take one or more statements and combine them to produce one statement. Once we have a set of axioms and a set of rules of inference, we say that a statement is true (i.e. is a theorem) if there is some way to repeatedly apply the rules of inference to the axioms to obtain the statement in consideration. This means that the truth of a statement depends not only on the axioms but also on the rules of inference. For our purposes, these rules of inference end up being those of propositional logic, which conform to our intuitive ideas of what "logical" means. However it is possible (and interesting!) to see what happens if different rules are used. We recommend the book *Gödel, Escher, Bach* by Douglas Hofstadter ([2]) to those interested in this kind of mathematics. It gives a great introduction to logic and draws interesting correspondences to other fields including art and music. It also introduces Gödel's incompleteness theorems, which in Mandar's opinion are two of the deepest results in all of mathematics.

5 Notation

For the purposes of this class, it is okay to think about sets intuitively (under the assumption that set theory works without paradoxes [which it does under the system of ZF set theory]).

Let A and B be sets. Below is a list of most of the notation we will be using throughout the course:

- $\{1, 2, 3\}$ is a set containing 1, 2, and 3.
- $\emptyset$ denotes the empty set.
- $a \in A$ means that a is a *member* of the set A.
- The **union** of A and B is the set containing those elements in either A or B. We denote it by $A \cup B$.
- The **intersection** of A and B is the set of elements common to A and B. We denote it by $A \cap B$.
- The **power set** of A is the set of all subsets of A. We denote it by $\mathcal{P}(A)$.
- $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ refer to the sets of natural, rational, and real numbers respectively.
- $\forall$, $\exists$, $\implies$, are shorthand for *for all*, *there exists*, and *implies*.

6 The Peano Axioms: Formalizing $\mathbb{N}$

6.1 The Peano Axioms

We will now define the set of natural numbers, $\mathbb{N}$, via the following 9 axioms. These axioms are known as the **Peano Axioms**. The first 4 axioms define equality on the set $\mathbb{N}$.

Axiom 1: For every $x \in \mathbb{N}$, we have $x = x$. (*reflexivity*)

Axiom 2: For every $x, y \in \mathbb{N}$, if $x = y$ then $y = x$. (*symmetry*)

Axiom 3: For every $x, y, z \in \mathbb{N}$, if $x = y$ and $y = z$ then $x = z$. (*transitivity*)

Axiom 4: For all x and y, if $x \in \mathbb{N}$ and $x = y$ then $y \in \mathbb{N}$ (*closure of equality*)

The remaining 5 axioms define the set $\mathbb{N}$.

Axiom 5: 0 is a natural number, i.e. $0 \in \mathbb{N}$

Axiom 6: We denote the *successor* of a number x by $S(x)$. The second Peano axiom states that if $x \in \mathbb{N}$, then $S(x) \in \mathbb{N}$, i.e. the successor of a natural number is a natural number.

Axiom 7: There is no $x \in \mathbb{N}$ such that $S(x) = 0$. In other words, 0 is not the successor of any natural number.

Axiom 8: For all $x, y \in \mathbb{N}$, if $S(x) = S(y)$ then $x = y$.

Notice that if we only had axioms 1 to 7, then the set $\{0, 1\}$ with $S(0) = 1$ and $S(1) = 1$ would be a perfectly valid candidate for $\mathbb{N}$. This is clearly not what we want, and axiom 8 ensures that.

Axiom 9: Let $P(x)$ be a statement about the natural number x. If:

- $P(0)$ is true
- Whenever $P(n)$ is true for some $n \in \mathbb{N}$, $P(S(n))$ is also true

then $P(x)$ is true for all $x \in \mathbb{N}$. This axiom is also known as the Principle of Mathematical Induction, and is a very useful tool for proving mathematical statements.[4]

As shorthand, we denote $S(0) = 1$, $S(S(0)) = 2$, $S(S(S(0))) = 3$, and so on, as to retain the numerals we are familiar with. If you feel inclined, however, denote these in any way you can think of!

6.2 Arithmetic on $\mathbb{N}$

We have defined the set $\mathbb{N}$, but we still have no way of working with the elements of $\mathbb{N}$. In this section we will formalize, using only the above 9 axioms, the two basic operations that can be carried out on natural numbers: addition and multiplication.

6.2.1 Addition

We define addition recursively:

Definition:

- For any $a \in \mathbb{N}$, $a + 0 = a$
- For any $a, b \in \mathbb{N}$, $a + S(b) = S(a + b)$

Example: Let us try to show that $1 + 2 = 3$.

$$\begin{aligned} 1 + 2 &= 1 + S(1) \\ &= S(1 + 1) \\ &= S(1 + S(0)) \\ &= S(S(1 + 0)) \\ &= S(S(1)) \\ &= S(2) \\ &= 3 \end{aligned}$$

Phew! Thank goodness...

6.2.2 Multiplication

Multiplication is defined in a way which should be rather familiar to you, perhaps bringing back memories of counting on your fingers in grade school. We define multiplication in terms of repeated addition. Again, our definition is recursive.

Definition:

- For any $a \in \mathbb{N}$, $a \cdot 0 = 0$

[4] This axiom is often stated in an alternative manner using the notion of an "inductive set". For the purposes of this class we won't be going over that, but in case you are interested [5] states axiom 9 based on inductive sets.

- For any $a, b \in \mathbb{N}$, $a \cdot S(b) = a + (a \cdot b)$

Example: We show that 2 times 2 equals 4.

$$
\begin{aligned}
2 \cdot 2 &= 2 \cdot S(1) \\
&= 2 + (2 \cdot 1) \\
&= 2 + (2 \cdot S(0)) \\
&= 2 + (2 + (2 \cdot 0)) \\
&= 2 + (2 + 0) \\
&= 2 + 2 \\
&= 2 + S(1) \\
&= S(2 + 1) \\
&= S(2 + S(0)) \\
&= S(S(2 + 0)) \\
&= S(S(2)) \\
&= S(3) \\
&= 4
\end{aligned}
$$

6.3 Ordering the Natural Numbers

Definition: For any $a, b \in \mathbb{N}$, we say that $a \leq b$ if there exists some $s \in \mathbb{N}$ such that $a + s = b$. If we can find $s \neq 0$ then $a < b$.

6.4 More Information

A more thorough explanation of the formalization can be found at [5]. We encourage you to take a look at it.

7 Some things to think about

Show that:

- $3 \cdot 2 = 6$
- $a \cdot 1 = a$.
- $a \cdot b = b \cdot a$.
- $a + b = b + a$
- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- $a + (b + c) = (a + b) + c$
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

References

[1] Bamber Gascoigne. History of counting systems and numerals. `http://www.historyworld.net/wrldhis/PlainTextHistories.asp?historyid=ab34`. Accessed 2020-07-07.

[2] Douglas Hofstadter. *Gödel, Escher, Bach: An Eternal Golden Braid*. Harvester studies in cognitive science. Penguin, 2000.

[3] Tony Lian. Fundamentals of zermelo-frenkel set theory. https://math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Lian.pdf. Accessed 2020-07-07.

[4] New World Encyclopedia contributors. Number. https://www.newworldencyclopedia.org/entry/Number. Accessed 2020-07-07. Section 4 ("History") is relevant for our purposes.

[5] Roberto Pelayo. Chapter 7: The peano axioms. http://www2.hawaii.edu/%7Erobertop/Courses/TMP/7_Peano_Axioms.pdf. Accessed 2020-07-07.