1 Solution to the Problem from Last Time

Recall that at the end of last session, we considered a set of five figures and asked ourselves which pairs show a "sub-symmetry – parent symmetry" relation. As a reminder, below are the five figures (in the simplified version):

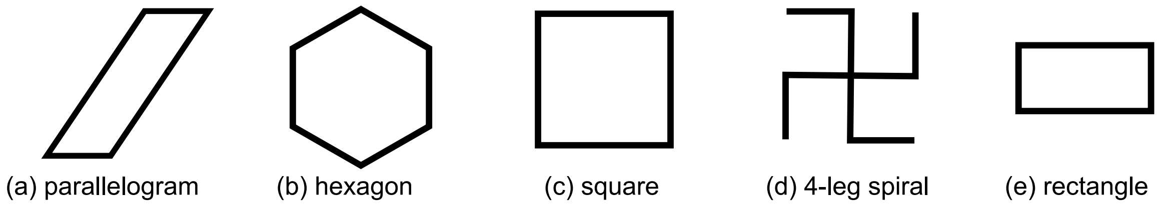
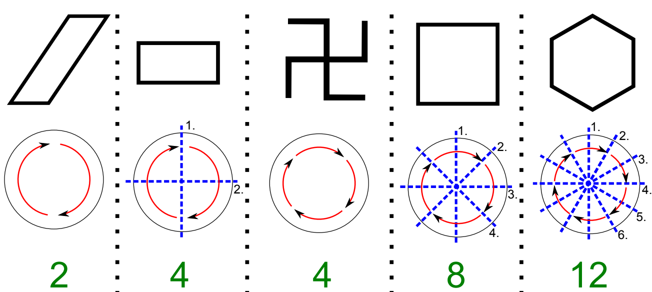
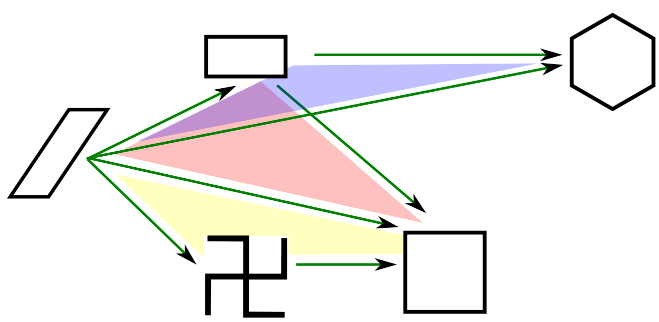


Figure 1: Five figures from the question last time.

To solve the problem, we first check the symmetry actions that are present in each figure, which are shown in Fig. 2(a). From this, we can check whether the symmetry actions of one figure is contained entirely in another. This allows us to find out all sub-symmetry – parent symmetry pairs, which are shown in Fig. 2(b), with arrows drawn from a sub-symmetry to a parent symmetry. Two figures that are not connected by an arrow is not related directly by the sub-symmetry relation.



(a) Symmetries of the five figures in Fig. 1.



(b) Sub-symmetry relations among the five figures in Fig. 1.

Figure 2: Solution to the question last time.

2 Properties of the Sub-symmetry Relation

Looking at Fig. 2, we can identify some general features of the sub-symmetry relation. First, observe that the arrows generally form triangles, which says that if A is a sub-symmetry of B and B is a sub-symmetry of C, then A is a sub-symmetry of C. If we go back to the definition of sub-symmetry, we can check that this property, called transitivity, indeed follows from the definition. To summarize:

> *Transitivity of the sub-symmetry relation:*
>
> If A is a sub-symmetry of B and B is a sub-symmetry of C, then A is a sub-symmetry of C.

Next, last time we saw that if an object A has a higher symmetry than another object B, then A has more symmetry actions than B. But if we look at the pairs that form a sub-symmetry relation in the above, we see that a *stronger* condition holds: When A is a sub-symmetry of B, the number of symmetry actions in A always divide the number of symmetry actions in B (the number of symmetry actions for each object is shown at the bottom of Fig. 2(a)). This again is a general property and is called the Lagrange's theorem, named after the French mathematician Joseph Louis Lagrange (1736 - 1813):

> *Lagrange's Theorem:*
>
> If A is a sub-symmetry of B, then the number of symmetry actions in A divides the number of symmetry actions in B.

Note that the two theorems above are somewhat clumsy to state. That's because we state everything in words (imagine writing "A equals to B" instead of $A = B$ everytime).[1]

[1] This is indeed how mathematics texts are written in the Medieval Ages. Back then, mathematical expressions are written in full sentences without the use of symbols like $\sqrt{\ }$ and $=$. How mathematics notations evolve in time is, by itself, an interesting subject.

We would have stated the same theorems much more compactly if we introduce two notations: first, use the symbol $\subset$ to denote sub-symmetry (so that $A \subset B$ means A is a sub-symmetry of B) as in last time; second, use the symbol $|\cdot|$ to denotes symmetry action counts (so that $|A|$ is the number of symmetry actions in A). Then, the two theorems can be restated as:

Transitivity of the sub-symmetry relation:

If $A \subset B$ and $B \subset C$, then $A \subset C$.

Lagrange's Theorem:

If $A \subset B$, then $|A|$ divides $|B|$.

Incidentally, this form of the Lagrange's theorem shows that the symbol $\subset$ is indeed a close analog of the familiar $<$ sign, since $a < b$ and $b < c$ also implies that $a < c$.

Lagrange's theorem is not obvious from the definition of sub-symmetry. We will gain more insights into how it works below....

3 Generating Sub-symmetries of a Given Figure.

Now that we understand how to compare the symmetries between two figures, we moved on to a different but related question. Suppose we are given a certain figure, how can we generate new figures that exhibit its sub-symmetries?

We have already seen one way how this can be done, namely, by distorting the original figure. For example, stretching a square will produce a rectangle, and tilting a rectangle will produce a parallelogram, each being a sub-symmetry of its predecessor (Fig. 3).

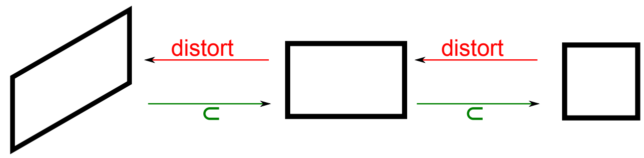
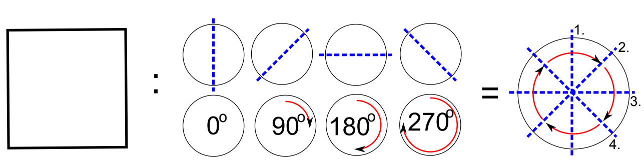


Figure 3: Generate sub-symmetries of a figure via distortion.

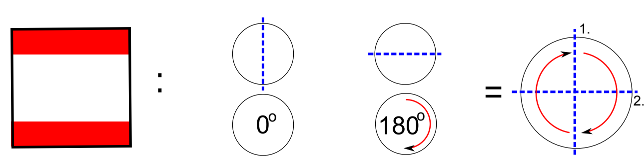
However, sometimes it is difficult to generate a known sub-symmetry relation this way. For example, we know the 4-leg spiral is a sub-symmetry of the square. But it is difficult to find a "natural" distortion of the square that exhibits the symmetry of the 4-leg spiral (Fig. 4).

Figure 4: Sub-symmetry relations that are difficult to realize via distortion.

Here we introduced another method that can be used to generate a figure that exhibits a sub-symmetry of the original figure. The method is simple: start with the original figure and highlight a portion of it. Then, the highlighted figure will exhibit a sub-symmetry of the original figure, since not all symmetry actions of the original figure will keep the highlighted feature unchanged. For example, consider highlighting the top and bottom edges of a square. The original square has eight symmetry actions (listed in Fig. 5(a)), while the highlighted square has only four (listed in Fig. 5(b)).



(a) Symmetry actions of the original square



(b) Symmetry actions of the highlighted square

Figure 5: The original square vs. the highlighted square.

There are, of course, many ways of highlighting the square. But many of them will share the same symmetry. For example consider Fig. 6. It is not difficult to see that (a) and (b) exhibit the same symmetry. Moreover, the symmetry exhibited by (c) is also essentially the same as that of (a) and (b), as long as we look at the figure at a 45°-angle tilt. In analogous manner, we see that the symmetry exhibited by (e) and (f) are essentially the same.

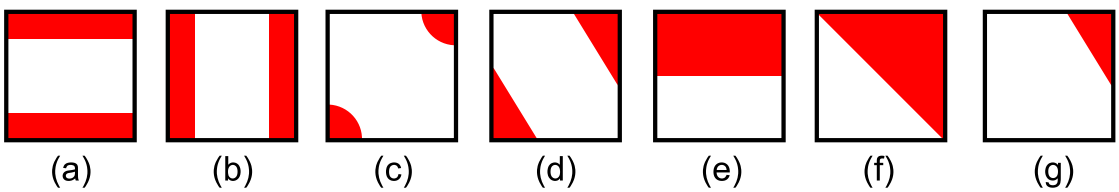


Figure 6: Several ways of highlighting the square.

Looking at Fig. 6(a) and Fig. 6(b) more closely, we see that they are actually closely related: one is the 90° rotation of the other. In fact, more can be said: the two are related by *any* of the four symmetry actions that are present in the original (without highlight) square but absent after the highlight! See Fig. 7 for illustration.

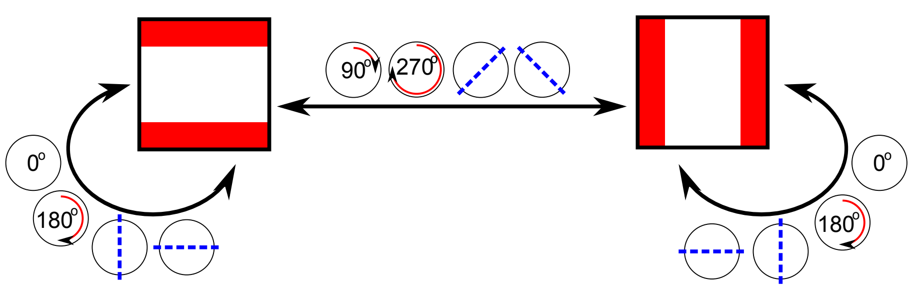


Figure 7: Two related ways of highlighting the square and their relation.

Similar situation also occurs for the highlighting pattern in Fig. 6(d). This time the symmetry of the highlighted pattern is reduced to 0° and 180° rotations, and there are four related ways of highlighting. Again, we see that the symmetry actions that are present in the square but absent in the highlighted pattern show up as actions that connect between the related highlighting patterns (Fig. 8).

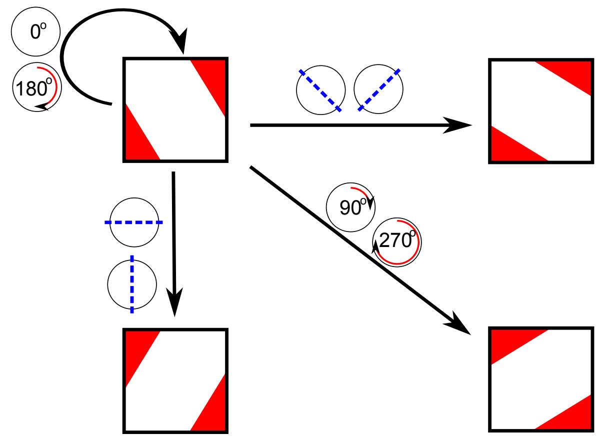


Figure 8: Two related ways of highlighting the square and their relation.

Furthermore, we see that the number of actions that connect between two related highlighting always equal to the number of symmetry actions that hold the highlighted square unchanged. This suggests how Lagrange's theorem works and give a meaning to the unspecified quotient in it.

Putting this quotient back, Lagrange's theorem becomes (see Fig. 9 for illustration):

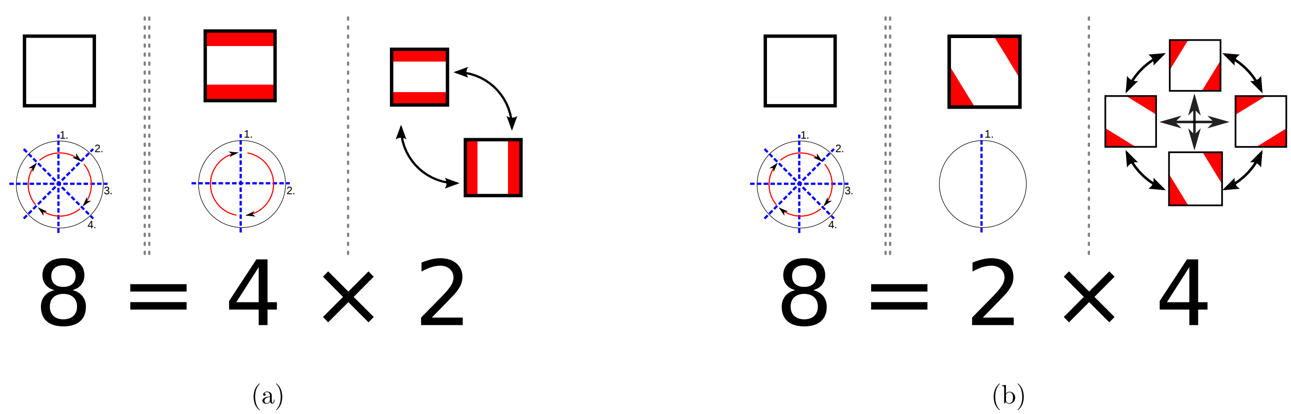


Figure 9: Illustrations of the Counting Formula.

The Counting formula for sub-symmetry:

$$\begin{array}{rl} & \text{number of symmetry actions in a sub-symmetry} \\ \times & \text{number of related figures} \\ \hline = & \text{number of symmetry actions in the parent symmetry} \end{array}$$

It should be noted that the same idea works for sub-symmetry that are generated from distortion. For example, there are two directions to stretch a square into a rectangle, and these two are related by the symmetry actions that are present in the square but absent in the rectangle.

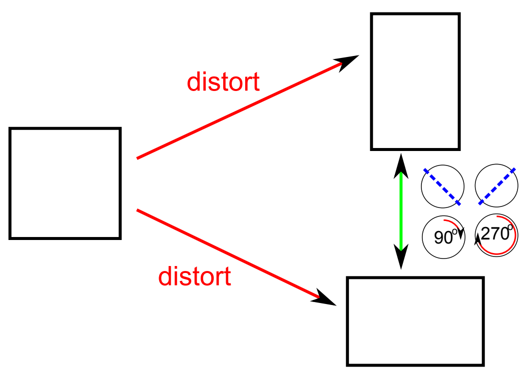


Figure 10: Two related rectangles obtained from distorting the same square in two ways.

But why should the number of symmetry actions that connect between two related shapes be the same as the number of symmetry actions that keeps the highlighted pattern unchanged? In a sense, this can be considered as a consequence of rule 2 about symmetry that we learned back in session 1.

Recall that rule 2 says that two symmetry actions performed in succession will produce yet another symmetry action. Given a highlighted figure and *one* particular symmetry action (say r) that converts it to a related figure, we can generate *all* the symmetry actions that produce the same conversion by composing a symmetry action that keep the highlighted pattern unchanged with r (see Fig. 11 for illustrations).

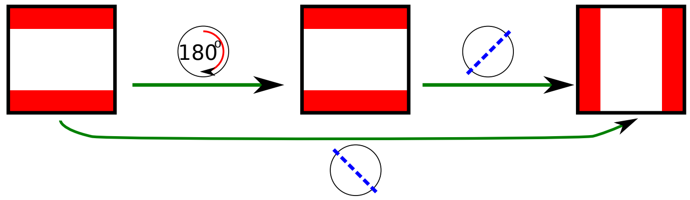


Figure 11: Illustrations of the Counting Formula.

By considering different ways of highlighting the square, we can find all the distinct sub-symmetries of the square. The results is shown in Fig. 12, where I show the sub-symmetry alongside with one set of related highlighting patterns. From Fig. 12, we see that there a square has precisely five distinct types of sub-symmetries, and these form a net of sub-symmetry relations among themselves, as shown in Fig. 13.

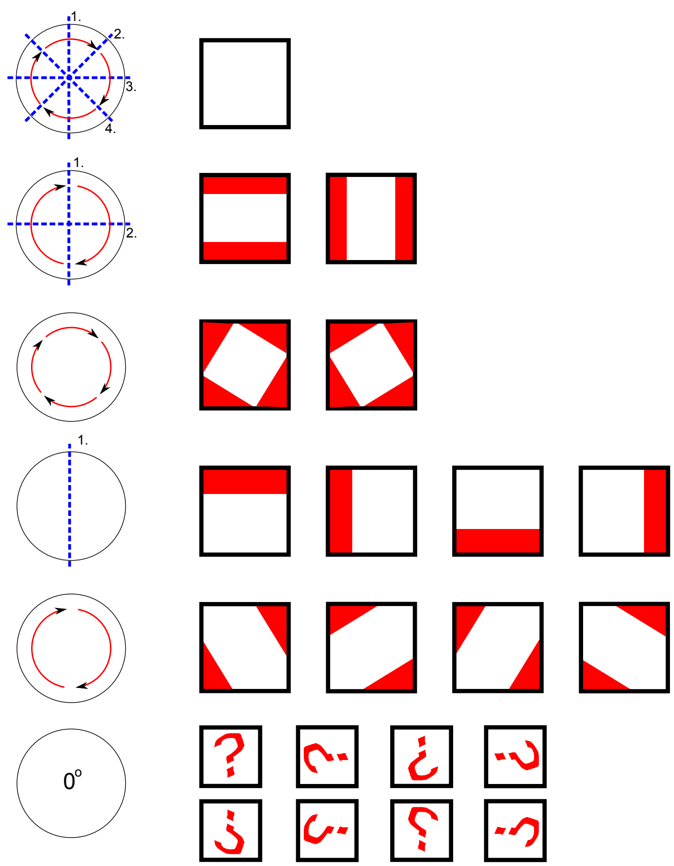


Figure 12: Distinct sub-symmetries of the square.

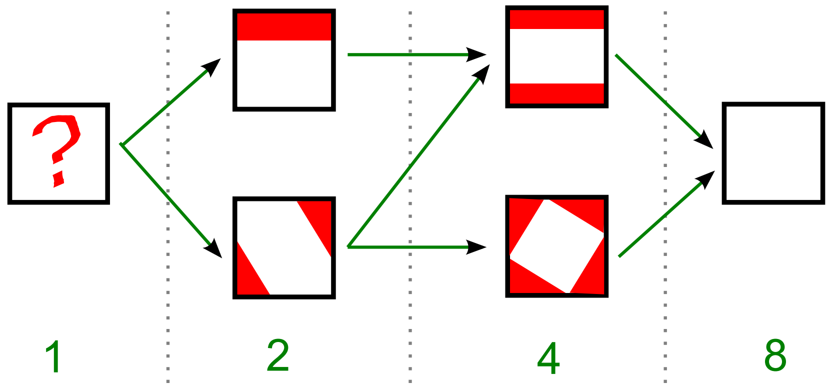


Figure 13: Distinct sub-symmetries of the square.

Now, your turn: can you use the highlighting method to find all the distinct sub-symmetries of a regular hexagon?