Splash! 2007
Combinatorics Problem Solving

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11-18-07

1 Addition and Multiplication

The Addition Principle
If event A has a possibilities and event B has b other possibilities, then the event of either A or B has $a + b$ possibilities.

The Multiplication Principle
If *independent* events A and B each have a and b possibilities, respectively, then the event of A happening, then B, has ab possibilites.

Problem 1. How many squares are contained within a 10×10 grid of points?

Problem 2. For how many pairs of consecutive integers in 1000, 1001, ... 2000 is no carrying required for addition?

Problem 3. How many zeroes are at the end of 86!?

Problem 4. How many numbers can be formed from the rearrangements of the digits of 40528?

2 Permutations and Combinations

Permutations - number of ways to choose k from set of n and arranging them (order matters!)

$$_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

Combinations - number of ways to choose k from set of n (no order)

$$\binom{n}{k} =_n C_k = \frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots(1)} = \frac{n!}{(n-k+1)!k!}$$

Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0}\binom{1}{1}$$
$$\binom{2}{0}\binom{2}{1}\binom{2}{2}$$
$$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$$

Combinatorial Identities

- Binomial Theorem
- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$
- $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$
- $\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n} = 0$
- $\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+m}{m} = \binom{n+m+1}{m}$
- $\binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \cdots + \binom{m}{k}\binom{n}{0} = \binom{m+n}{k}$

Problem 1. How many ways can you rearrange the letters of MASSACHUSSETTS?

Problem 2. How many ways can you arrange 3 physics books, 5 math books, and 2 chemistry books on a shelf if the books must remain together by subject?

Problem 3. I have a 3-term sequence lock, but I have forgotten one of the numbers. At most how many tries will I need to unlock my lock?

Problem 4. Find the number of ways to arrange King Arthur's 12 knights in a row of chairs. What if it is a circle of chairs?

Problem 5. How many numbers between 0 and 1 can be expressed as a fraction such that the product of the numerator and denominator is 25!?

Problem 6. Fifteen points are on a circle and each pair is connected by a line segment. How many points of intersetion inside the circle are there?

Problem 7. Billy Bug wanders around a grid, starting at (0,0) and only stepping 1 unit horizontally or vertically. What is the probability that he'll end up at (3,2) in at most 7 steps?

Problem 8. What is the constant term in $(x^2 + \frac{1}{x^2} - 7)^10$?

Problem 9. Find n if

$$\binom{n}{k} : \binom{n}{k+1} : \binom{n}{k+2} = 3 : 4 : 5.$$

Problem 10. How many ways are there to choose any number of squares on a 8×8 chessboard so that none are in the same row/column and no square exists to the Southwest of any other square?

Problem 11. I have 31 little brothers, 10 of whom are clones of each other. I want to take 10 of them to the movie theater today, but I cannot tell between the clones. How many ways can I do so?

3 Principle of Inclusion-Exclusion

Problem 1. MIT has 85 seniors, each of whom plays on at least one of the three varsity sports teams: football, baseball, and tennis. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and tennis teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and tennis teams. Compute the number of seniors playing all three sports, given that twice this number are members of the tennis team.

Problem 2. King Arthur wants to send 3 of his twelve knights on quest to save a princess. Assuming the 12 knights are sitting at the round table, what is the probability that none of the three are sitting next to each other?

Problem 3. Find the number of primes less than 120?

Problem 4. What is that probability that a randomly chosen rectangle on an 8×8 checkerboard is not a square?

Problem 5. All the squares in a 5×5 checkerboard are gray except two that are yellow. How many different arrangements are there, if two that are equivalent by rotation are the same?

Problem 6. How many ways can I distribute 5 different candy bars among 3 kids, where each kid gets at least one bar?

Problem 7. I just bought 7 a's, 6 m's, and 5 w's from Dictionopolis. How many ways can I arrange the letters so that at least 3 aw's appear?

Problem 8. Four twins are occupying the 8 seats in the front row of a movie theater. How many ways can they sit so that no one is next to his/her twin?

Problem 9. A dot is marked at each vertex of a triangle ABC. Then, 2, 3, and 7 more dots are marked on the sides AB, BC, and CA, respectively. How many triangles have their vertices at these dots?

Problem 10. How many solutions are there to $x_1 + x_2 + x_3 + x_4 = 30$ if no x_i is greater than 12?

4 Bijections

The **Balls and Bins** model.

Problem 1. How many ways can you distribute 10 pieces of candy among 3 kids if each kid must receive at least 1 piece?

Problem 2. There are b boys and g girls in this class. How many ways can we sit in a circle so that no two of the minority group are next to each other?

Problem 3. Five dice are rolled. What is the probability that the sum of the 5 numbers is 14?

Problem 4. A triangular grid is formed by dividing an equilateral triangle of side length n into smaller equilateral triangles of side length 1. How many parallelograms can you find?

Problem 5. How many ways can I put 11 balls in 7 bins if 2 of the bins must remain empty?

Problem 6. Find the sum of all products xyz where x, y, and z are solutions to $x + y + z = 17$.

Problem 7. Delegates from 18 countries are meeting in a building that has 8 flagpoles. How many ways can one arrange the flags on the poles if no pole may remain empty? If pole may remain empty?

5 Pigeonhole Principle

Problem 1. Twenty five boys and twenty five girls sit around a table. Prove that it is always possible to find a person both of whose neighbors are girls.

Problem 2. A small town has 20 children. Each pair of children has at least one grandfather in common, and each child has two grandfathers. Show that a grandfather exists with at least 14 grandchildren.

Problem 3. Prove that if 5 points are in a square if side length $\sqrt{2}$, then there exist two points at most distance 1 from each other.

Problem 4. King Arthur's 12 knights hail from n different provinces of England. If two knights are from the same province, then the knights on their left are from two different provinces. Find the minimum value of n by proving that there cannot be more than n^2 knights.

Problem 5. The "Math Lotto" betting card is a 66 table. The gambler is to mark 6 of the 36 slots in the card. The official result is published with 6 slots chosen as the "LOSING SLOTS". The gambler wins if he doesn't pick any losing slot. Prove that it is possible to fill out 9 betting cards so that at least one of them is a winner. Describe the markings on the 9 cards. Prove that 8 cards are not sufficient to ensure a win.

Problem 6. Prove that among any 10 points located in a circle of diameter 5, there exist 2 at a distance of at most 2 from each other.

6 Probability

Problem 1. Each of two boxes has black and white marbles, and there are a total of 25 marbles. Two marbles are randomly drawn, one from each box, and the probability they are both black is $\frac{27}{50}$. What's the probability that both are white?

Problem 2. A jar has 499 fair pennies and a fake penny with heads on both sides. I pick a penny at random and get all heads on my first 9 tries. What is the probability that I picked the fake penny?

Problem 3. In the 2003 World Series, the Marlins won over the Yankees, 4-to-2. Suppose that you were on Mars at that time with no contact with the Earth, but you know that the Marlins have a $\frac{2}{3}$ chance of winning any game. When you comeback and hear that the series lasted 6 games, what is the probability that you think the Marlins won?