2 Lecture 2: Point sets and set operations

2.1 Review

Last time we discussed three important concepts: functions, operations, and sets, all of which are what we could call "mathematical objects". In particular we looked at operations of numbers, shapes, and functions. We also looked at explicitly defined sets, like $\{2, 3, 5\}$, $\mathbb{N}$, and $\mathbb{Z}$. Our goal today is to study sets defined in a different way, as well as operations of sets.

Recall that a function f is an input-output relation between numbers that can be viewed in a few equivalent ways: as a table, as a graph, or as an algebraic expression. We do not allow functions to map an input to more than one output, but they can map multiple inputs to the same output.

For example, the function $f : x \to x^2$ takes a number x and squares it. We can write this as $f(x) = x^2$. Later in this class we will be more formal with functions, and specify the sets that the function works with.

After functions, we talked about operations, which take in any object (not just numbers) and return another. We looked at an operation that added two functions, returning a new function. If f and g are functions, then $h = M(f, g) = f + g$. That means for each input x, the function $h(x) + f(x) + g(x)$.

If we wish, we can even view the relations $<, >, =, \leq, \geq$ as an operation that outputs "True" or "False".

Finally, we looked at some sets, which are an object that can contain other objects. These sets are denoted by curly braces $\{\}$. The objects they contain are written between the braces but that the order written does not matter.

The objects that a set contains are called its elements. If a is an element of the set A we write $a \in A$. If b is not an element of A, we write $b \notin A$. We looked at some specific sets like $\mathbb{N}$ and $\mathbb{Z}$, as well $\{2, 3, 5\}$ and $\varnothing$.

Example 8. Sets are allowed to contain other sets. A fabricated example is the set $\{\{0\}, \varnothing, \{x \in \mathbb{N} : \text{x is prime}\}\}$ which contains three unique sets.

2.2 Set definitions

Explicitly listing out each element of a set is not always the most effective way to define it, especially if there are an infinite number of elements in the set.

Usually its easier to define a rule or condition that can be used to determine if an object is an element of the set or not. These rules are formally the traits we consider in everyday examples, such as in the saying "a fruit has to have seeds to actually be considered a fruit. A mathematical example is that a natural number must only be divisible by itself and 1 in order to be a prime.

A bar $|$ or colon : is used to make these more implicit definition statements. The information following this separator is one or more conditions (or rules) that must be fulfilled for an object to be an element of that set.

For example, if we say "let A be the set of even integers" we could define A like so: $A = \{x \in \mathbb{Z} : \text{x is divisible by } 2\}$.

A useful example of this notation is in our definition of the rational numbers, $\mathbb{Q}$. We define $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}$ and $b \neq 0$. The letter Q comes from the word quotient. By definition, rational numbers can be expressed as a quotient a/b for a dividend a and divisor $b \neq 0$.

You may be aware of **irrational numbers** such as π and $\sqrt{2}$. The real numbers $\mathbb{R}$ contain both the rational numbers and the irrational numbers. However a formal construction of the real numbers is very technical and unfortunately is beyond the scope of this course. We can just use the intuitive definition that $\mathbb{R}$ contains any number we can think of.

"Construction" explains how to build one set out of another. For example, we could construct the integers from the natural numbers using subtraction.

2.3 Point sets

So far the discussion has not been too visual. Let's talk about sets of points using the implicit method of defining a set.

First, recall a point (in, say, two dimensions) is an ordered pair of the form (x_1, x_2). To be precise we need to stipulate what set x_1 and x_2 belong to! Let's say $x_1, x_2 \in \mathbb{Z}$. We also need to state what *coordinates* we are using to identity the point. The coordinates tell us what the grid on the graph looks like, and then the point (x_1, x_2) tells us to move x_1 tick marks in the first coordinate, and x_2 in the second. Usually we assume the grid is Cartesian, which is when the first coordinate is horizontal and the second is vertical. The name comes from René Descartes, a famous philosopher, mathematician, and scientist.

Again, we could manually define a point set, listing out each point.

Example 9. Consider a point set $P = \{(0,0), (0,1), (0,2), (2,0)\}$.

But more commonly, point sets are defined implicitly by a function.

Example 10.

1. The set of lines through the origin can be defined as $\{(x, mx) : x, m \in \mathbb{R}\}$.

2. The circle of radius r, at the origin, is the set $C_r = \{x_1^2 + x_2^2 = r \mid x_1, x_2 \in \mathbb{R}\}$. Notice that the labels x_1, x_2 here do not matter. We could have used x, y, or anything else.

3. A right triangle with hypotenuse $\sqrt{2}$ is given by $T = \{(x, y) : x + y \leq 1, x, y \geq 0\}$. Other congruent triangles can be defined in similar ways.

Venn diagrams are a pictographic tool that can be seen as an abstraction of these point sets. We draw a shape that can be anything and associate it with a set. As we'll see shortly, set operations can be viewed as geometric relations between these shapes.