1 Lecture 1: functions, sets, and operations

1.1 Functions

The reader may have seen the concept of a *function* in math class before. This section will consist of a short review of this idea as a refresher. If functions are an unfamiliar idea, consider doing the exercises at the end.

Functions are very important because they appear often both in mathematical and physical scenarios. In short, a function is a mathematical object that turns one number into another according to some rule. We have great flexibility in choosing this rule. For instance, a valid rule is "double the number", turning 4 into 8, 5 into 10, etc., or maybe the rule is to do nothing, returning the number that we began with. This rule is sometimes called an "input-output relation". The number we begin with is the input, and the number we end with is the output. **The only restriction on the rule is that each input must have exactly one output.** This does not mean two inputs cannot have the same outputs. A valid rule is then "return the number 6, for any input."

A function is *defined* by the rule it applies to numbers. However this rule can be expressed in many ways. Right now we will exemplify three different representations: tabular (meaning tables), graphical (meaning graphs), and algebraic (meaning variables). Later in the course we will explore how functions can be described with sets.

1. The tabular representation of a function is built on a table of input output values. For instance, if the rule is "double the input", a portion of that table is given by

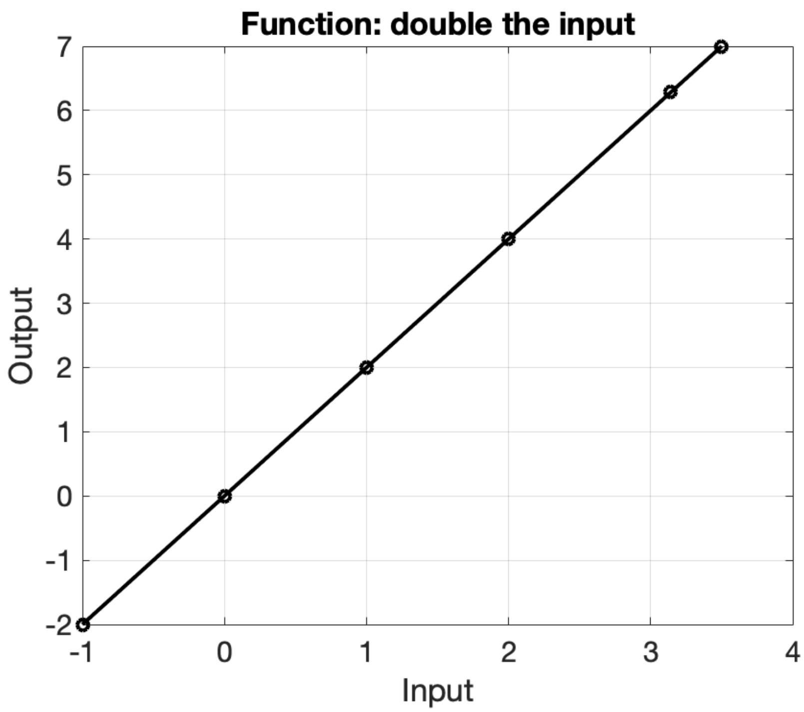
Input	**Output**
-1	-2
0	0
1	2
2	4
π	2π
3.5	7

Table 1: Tabular representation of a function which doubles the input.

   Of course, this function works with any number we can think of, so to be complete, would need an infinite number of rows. This isn't possible to fit here, so let's consider the next representation.

2. The graphical form of a function is obtained by plotting all the points from the table. The horizontal (X) axis is the input and the vertical (Y) axis is the output. Again, we are unable to plot all the infinite number of points, but we can plot some of them and then connect the dots.

3. The graph is a useful visual tool, but it is not perfect. We are still often unable to plot the entire function. A more efficient and complete represen-



tation is the algebraic one. To obtain this, we need to write the rule down in terms of variables. If the rule is "double the input", we can call the input x. Then the output is found by doubling x, which is $2x$. Therefore, the function takes x and turns it into $2x$. Let's represent the **action** of the function, which is a fancy way of saying "what it does to the input", by the arrow "$\to$". So our function does this: $x \to 2x$. Finally, let's give this function a name, such as f, for function.

Putting the pieces together, we have defined a function f by the rule $f : x \to 2x$. In words, this says "f takes x and turns it into $2x$".

When this function is *evaluated*, which means it is given a number to apply its rule to, we use a special notation: Say we want to evaluate our function f at some point x, which is a variable, then we write $f(x)$. This notation can be used to define functions instead of using the arrow. For our example, we would write $f(x) = 2x$. If we want to triple the input, we would write $f(x) = 3x$. Or if we have another function g that squares its input z, we could write $g(z) = z^2$.

Like with variables, the exact symbol we use for the function or its input can be whatever we wish! Instead of writing $g(z) = z^2$ one could write $g(\phi) = \phi^2$, where ϕ is the Greek letter "phi". But in each case the function g is the same, because it has the same rule.

1.1.1 Practice problems.

1. Express the function that turns a positive number into its square root in all three forms. For the inputs, use 0, 1, 4, 9 and 16.

2. Log into www.desmos.com/calculator and plot the function $f(x) = 2x$. This tool is very useful for plotting functions. Spend a few minutes plotting whatever functions you like. For example, try $f(x) = 1/x$ and $f(x) = -x^2 + 5$.

1.2 Introduction to sets

Many concepts are studied through comparison, especially ones one may not associate with math. This is often used for organizational purposes and has many benefits. Alphabetized files in a cabinet make finding a particular file easier, and having movie genres in Netflix makes it easier to find a comedy. To understand how long a distance is, we might compare it with a familiar reference distance, like a ruler.

Without these organizational concepts, life can get pretty difficult. Just imagine if instead of using ascending numbers, we put random symbols on houses. Or if google gave results with no relation to your search query. It's no mistake that some modern computer models are given the job of finding similarities in datasets and grouping similar data together.

An organizational system requires that we conceptually group things together that mutually share some characteristic, which can be anything we wish. Not only does this grouping make things like finding a movie an easier, more efficient process, but also allows us to *define and discuss new concepts* in a concrete manner, like the "kilometer" or the collection of "mystery novels." The notion of a trait or characteristic emerges as a result of our comparison.

In this class, the main goal is to extend the ideas of compare, contrast, and "group together" to mathematical objects. Each collection of similar objects will be called a set and are denoted by curly braces $\{\ \}$. Everything between those braces is a member or **element** of the set. If an object is an element of a set, we will use the symbol "$\in$" to denote this. If an object is not element of the set, we draw a line through "$\in$" like we do with "$=$" and "$\neq$", giving the symbol $\notin$.

Sets with multiple elements are often written with the elements separated by commas, but the order of the elements does not matter.

Example 1. Just like we might say "let x be a variable with the value 1" and write $x = 1$, we can say "let A be the set containing the number 1" and write $A = \{1\}$. Then we can say $1 \in A$ and read this as "1 is an element of A."

If A is instead the set containing the first three prime numbers, we could write $A = \{2, 3, 5\}$ (or $A = \{3, 2, 5\} = \{5, 3, 2\} = \{5, 2, 3\} = \{3, 5, 2\} = \{2, 5, 3\}$ because the order does not matter).

These sets provide us an intuitive and powerful language to describe mathematical properties. They can easily have an infinite number of elements! Instead of writing down all of these elements, we use ellipses "$\ldots$" to let the reader fill in the gaps.

Example 2. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}$ be the set of integers. Then, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ and $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Note that if $x \in \mathbb{N}$ then $x \in \mathbb{Z}$ is always a true statement. However, if $x \in \mathbb{Z}$ then $x \in \mathbb{N}$ is not always true. For instance, $0 \in \mathbb{Z}$ but $0 \notin \mathbb{N}$.

Besides a finite and infinite number, sets can have no elements. This last example is so important that it receives its own special symbol.

Example 3. The set with no elements is called the **empty set** and is denoted $\varnothing$.

A term one often hears in mathematics are variants of the word *general* (generalize, generalization, etc.). Set theory is useful to us because it allows us to make general statements, as opposed to specific ones. To give an example that contrasts these two ideas, consider these two statements.

1. The area of a unit-radius circle is π.
2. The of a circle with radius r is πr^2.

The second statement is more general because it includes the first statement as a special case $r = 1$. General statements are useful because they can be applied in more places. As we've said, set theory is useful due to its elegant framework for categorizing mathematical objects that have common properties. Beyond that, it helps us make pose and establish the validity of general statements: rather than show something applies to a single object, we can show it applies to a whole set of objects. In the context of set theory, we might rephrase the statements above like so.

Example 4. If c is an element of the set of all circles C and has a radius r, then the area of c is πr^2. In particular, if c has radius 1, then the area of c is π.

More abstractly, we could view the area formula as an operation that maps one set to another. In particular it transforms the *set* of all circles to the *set* of positive real numbers. This abstract viewpoint may seem a little unfamiliar at this point. It was only expressed here to illustrate where the ideas we will discuss later will take us.

1.3 Mathematical objects and operations

Before diving further into set-theoretic notation, let's pause to consider what we mean by "mathematical object." Anything you encounter in math is a mathematical object. Common examples are numbers, shapes, graphs, points, and functions.

An **operation** is an object that takes one or more other objects, known here as **operands** or sometimes arguments, and produces a new object with them. This is basically like a function, except that here the arguments can be anything, not just numbers. Still, we can use the perspective of functions: let $C = f(A, B)$. Then f is an operation on B and B that produces C. A, B, and C are any object we wish, and f is a rule that explains how A and B are combined to produce C.

Example 5. Many operations are probably quite familiar to the reader already. For very common operations we use shorthand notation, the first one for most people being addition, $f = +$. Instead of writing $+(1, 2) = 3$, we write $1+2 = 3$. The same shorthand is used for all arithmetic operations $+, -, \times, \div$.

Operations can be quite general, so let's look at some more examples, and at the same time, look at some sets of non-numeric objects to get a better sense of how sets are defined.

Example 6. When first exposed to functions, one is often taught that functions map numbers to other numbers: $f(x) = x^2$ maps $1 \to 1$, $2 \to 4$, $3 \to 9$, and so on.

But one can define operations on the function itself, which can produce numbers or other functions. For example, let $M(\cdot, \cdot)$ be the operation which takes two functions and produces their sum. That is, $M(f, g) = f + g$, and we keep in the back of our mind that f and g are functions of some variable.

For example, if $f(x) = 2x$ and $g(x) = x^2$, $h(x) = (M(f, g))(x) = (f+g)(x) = 2x + x^2$.

Example 7. Not all operations require two arguments. Some have more, and some only have one. Consider the increment operation, which is directly built into the hardware of practically all modern computers. This operation is defined as $I(x) = x + 1$ and acts like a counter. This operation is also common enough to receive its own shorthand, which is found in programming contexts. $I(x) = ++x$, so $++10 = 11, ++20429 = 20430$, and so on. Sometimes folks will have ++ following the number, $I(x) = x++$.

Saying it was built "directly" into the hardware here means it cannot be decomposed into other operations that the computer is capable of. An example of the other case, in some computers, is addition. In those computers addition is decomposed into a series of increments. So if the computer is asked to add 10+3, it increments 10 exactly 3 times: $++(++(++10)) = ++(++11) = ++12 = 13$.

This is essentially the "addition by counting" that children are often initially taught.