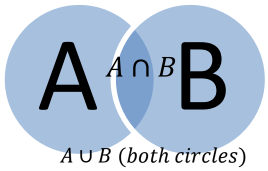
Preparation for Linear Algebra Class

A **set** is any collection of elements. For 2 sets A and B, their **union** $A \cup B$ is the set of all elements in either of them, and their **intersection** $A \cap B$ is the set of all elements in both of them.



Sets may be defined using the following notation: (| means "such that")

$$S = \{f(x) | x \text{ satisfies condition BLAH}\}$$

For example,

$$\{0,1,4,9,16, \ldots\} = \{x^2 | x \text{ is an integer}\}$$

$$\{1, -1\} = \{x | x^2 = 1\}$$

A **function** $f: S \to T$ takes each element from a set S and outputs an element from T. S is called the **domain**, and T the **range**. The most common examples are functions on the reals $\mathbb{R} \to \mathbb{R}$ (like $f(x) = x^2$, etc.). A function is **one-to-one** (bijective) is each element is sent to a different element (for example, $f(x) = 2x$ is one-to-one, but $f(x) = x^2$ is not because $f(1) = 1 = f(-1)$.) The **inverse** of a function f "undos" the operation (if $f(x) = x + 1$, then $f^{-1}(x) = x - 1$), but it is a function only if f is one-to-one. For example, over the positive reals, if $f(x) = x^2$, then $f^{-1}(x) = \sqrt{(x)}$, but this is not true over all reals because BOTH $u = \pm\sqrt{x}$ give $f(u) = x$. $fg(x)$ means the function defined by <u>first</u> carrying out g, <u>then</u> carrying out f, i.e. $f(g(x))$. As an exercise, convince yourself that $(fg)h = f(gh)$.

A **polynomial** is an expression constructed using variables, where only multiplication by constants, adding/subtracting, and raising to nonnegative powers is allowed, for example, $x^3 + 3x^2 + 3x + 1$.

Arithmetic can be defined for sets besides real numbers. When we take integers **modulo** a number n, we take the remainder of whatever result we get after division by n. In other words, we do not distinguish between numbers that are a multiple of n away from each other. For example modulo 3 (in $\mathbb{F}_3$ or $\mathbb{Z}/3\mathbb{Z}$),

$$1 + 2 = 0$$
$$2 \times 2 = 1$$
$$-1 = 2$$
$$2^{-1} = 2 \text{ (since 2x2=1)}$$

Note that in this example, addition and multiplication still satisfy the commutative, associative, and distributive laws. You can even find inverses for addition and multiplication, except the inverse of 0 is not defined. Any set with $+, \times$ defined, satisfying these rules, is called a **field**.

Summation notation

$\sum_{i=1}^{n} f(i)$ means $f(1) + f(2) + f(3) + \cdots + f(n)$. For example, $\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$. The sum may also be written with a condition on the bottom:

$$\sum_{1 \le i \le 5, i \ odd} i^2 = 1^2 + 3^2 + 5^2$$

Sum the expression to the right over all the cases where the bottom conditions are true.

See if you understand this identity:

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 + 2 \sum_{1 \le i < j \le n} x_i x_j$$

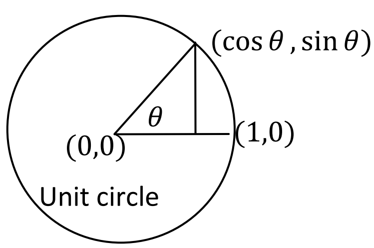
The sum on the right means "sum over all pairs (i,j) satisfying $1 \le i < j \le n$" For example, when $n = 3$ this says:

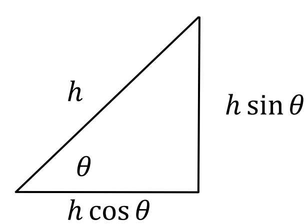
$$(x_1 + x_2 + x_3)^2 = (x_1^2 + x_2^2 + x_3^2) + 2(x_1 x_2 + x_1 x_3 + x_2 x_3)$$

(which is true by expanding the left-hand side.)

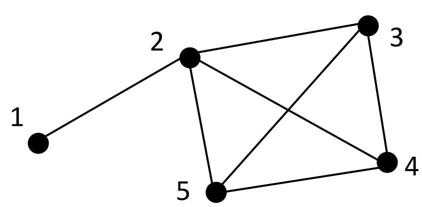
Trigonometry

The **sine** and **cosine** of an angle θ are the x and y coordinates when we rotate the point (1,0) by θ around the origin:





In combinatorics, a **graph** is a collection of vertices, some pairs of which can be connected by edges:



Only which pairs are connected is important; the shape is not. We may use graphs to represent relationship between objects, as in: There is a group of 5 people numbered 1, 2, 3, 4, 5. Represent the people as vertices, and connect two vertices if the corresponding people are friends. So above, people 2, 3, 4, and 5 are all mutual friends, but 1 is only friends with 2.

A $m \times n$ **matrix** is a table with m rows and n columns, filled with numbers. To add matrices (of the same dimensions), just add the corresponding entries. When multiplying by a scalar (constant), multiply every term by the scalar. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & -3 \end{bmatrix}, 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$