- Complete as many parts of the following two problems as you can. Do not worry if you cannot answer all of the parts; you are not necessarily under-qualified to take this course if you can't (and probably not over-qualified even if you can).
- Please document your thought process clearly. Your work is much more important to us than your final answer.
- Feel free to consult references or make use of software packages such as Mathematica, Maple, or Wolfram Alpha, as long as you clearly document which aids you used and where you used them.


## Part 1. Math

One of the things we will study in this class is the behavior of systems in which we treat time as a discrete variable. In particular, suppose we pick some initial $x_{0}$ with $-1<x_{0}<1$ and then generate $x_{1}, x_{2}$, and so on according to the rule (called a map)

$$
x_{n+1}=f\left(x_{n}\right)
$$

where

$$
f(x)=1-\alpha x^{2},
$$

and $\alpha$ is a parameter between 0 and 2 .
(a) A fixed point is a point $x^{*}$ such that $f\left(x^{*}\right)=x^{*}$. For example, when $\alpha=0,1$ is a fixed point. Find the fixed point $x_{\alpha}^{*}$ of our map as a function of $\alpha$.
(b) A fixed point is stable if, for small $\varepsilon, f\left(x^{*}+\varepsilon\right)$ is closer to $x^{*}$ than $x^{*}+\varepsilon$ is. In other words, points near the fixed point move toward the fixed point: $\left|f\left(x^{*}+\varepsilon\right)-x^{*}\right|<|\varepsilon|$. For which values of $\alpha$ is $x_{\alpha}^{*}$ stable? You may wish to use the fact that $f(x+\varepsilon) \approx$ $f(x)+\varepsilon f^{\prime}(x)$ for small $\varepsilon$.
(c) How does the map behave for other values of $\alpha$ ? Does it vary in a regular way as you increase $\alpha$, or does it do something else? Describe the behavior of the map as best as you can. For various values of $\alpha$, you may wish to examine the the behavior of the map for large $n$.

## Part 2. Physics

A particle of mass $m$ moves along the $x$-axis under the influence of some potential $V(x)$. You determine that the time $t$ at which the particle passes through a position $x$ is given by

$$
t(x)=2 t_{0} \ln \left(e^{x /\left(2 x_{0}\right)}+\sqrt{e^{x / x_{0}}-\alpha}\right)
$$

where $\alpha$ is a constant.
(a) Show that

$$
\frac{d t}{d x}=\frac{t_{0}}{x_{0}} \frac{e^{x /\left(2 x_{0}\right)}}{\sqrt{e^{x / x_{0}}-\alpha}}
$$

(b) Determine the particle's kinetic energy $K(x)$ as a function of $x$ and simplify your expression. You may wish to make use of the fact that $d y / d x=1 /(d x / d y)$.

Assume that the total energy of the particle is conserved. In other words, the sum of the kinetic energy $K$ and potential energy $V$ is the total energy $E$ :

$$
E=K(x)+V(x)
$$

(c) Explain why we are free to choose $E$ to be anything we want without changing the physics, as long as we make sure to set $V(x)=E-K(x)$.
(d) Choose $E$ such that for $x \rightarrow \infty$, the potential energy goes to zero. What is $V(x)$ given this choice of $E$ ?
(e) Determine $\alpha$ so that the particle passes through $x_{0}$ when $t=t_{0}$. Given this choice of $\alpha$, what are the kinetic energy and velocity of the particle at $x_{0}$ ?

