Splash! 2007
Geometry Problem Solving

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1 Angle Chasing

Sums of angles
Triangle Inequality: $a + b > c$
Pythagorean Theorem
Parallel lines and transversals

Problem 1. In a circle with center O, AD is a diameter, and AC is a chord. B is on AC such that $BO \perp AD$, and $\angle ABO = 60°$. Determine BC.

Problem 2. A semicircle with diameter AB is constructed inside square $ABCD$. E is on AD such that CE is tangent to the semicircle at F. Find the length CE.

Problem 3. R, V, S, and U are on a circle such that RS and UV meet at Q. RV and US extended meet at T outside the circle. $\triangle RST \cong \triangle UVT$. $\angle R = 36°$ and $\angle T = 42°$. What is $\angle RQV$?

Problem 4. On line SRT, semicircles are constructed with diameters SR and RT. PA is tangent to semicircle SAR and PB is tangent to TBR. Suppose $\widehat{AR} = a, \widehat{BR} = b, \widehat{AS} = c$, and $\widehat{BT} = d$. Show that $\angle APB = c + d$.

Problem 5. Quadrilateral $ABCD$ has $\angle ABD = \angle ACD = 60°$. Given that $\angle ADB = 90° - \frac{1}{2}\angle BDC$, prove that $\triangle ABC$ is isosceles.

Problem 6. A triangle has integral side lengths of 11, 17, and k. How many possible values are there for k?

Problem 7. In triangle ABC, D and E are on AB and AC respectively, so that $\angle AED$ is a right angle. Given that $AD = DB = 17$, $2AE = EC = 16$, and $DE \perp AC$, what is BC?

Problem 8. Rectangle $ABCD$ has a point P in it. Given $PA = 5$, $PB = 3$, and $PC = 10$, find PD.

2 Similarity

Similarity Theorems: SSS ˜; SAS ˜; AA ˜
Power of a Point

$$PC^2 = PA \cdot PB$$

Problem 1. E is on AB and D is on BC such that $AE : EB = 3 : 1$ and $CD : DA = 1 : 2$. What is $EF/FC + BF/FD$, if BD and CE meet at F?

Problem 2. If $AB \parallel CD \parallel EF$, then $1/AB + 1/EF = 1/CD$.

Problem 3. Trapezoid $ABCD$ has $AB||CD$ and $AB = 3CD$. Let the diagonals meet at E. Given that $BD = 15$, find BE.

Problem 4. Isosceles triangle ABC with vertex C has $\angle C = 108°$. Point M is inside ABC such that $\angle MCA = 23°$ and ∠MAC=7○. Find the the measuree of $\angle CMB$.

Problem 5. Trapezoid $ABCD$ has $AB||CD$. The diagonals meet at P. If $[APD] = 2[BPC]$, then what is $[ABP] : [APD]$?

3 Centers of Triangles

- **medians** stretch from vertex to midpoint of opposite side
  meet at the **centroid**
  intersect each other at a 2:1 ratio
- **angle bisectors** bisect vertex angles
  meet at the **incenter**, incircle has radius r
  Angle Bisector Thm $\frac{AB}{AC} = \frac{BD}{CD}$
- **perpendicular** bisectors are perpendicular to and bisect sides
  meet at the **circumcenter**, circumcircle has radius R
- **altitudes** stetch from vertex and are perpendicular to sides
  meet at the **orthocenter**

Problem 1. In ABC, AH is an altitude and BM is a median. $\angle A = 100°$ and $\angle B = 50°$. What is $\angle MHC$?

Problem 2. I is the incenter of ABC. If $AB = AC = 5$ and $BC = 8$, what is AI?

Problem 3. The median and angle bisector of ABC from B are BD and BE, respectively. Compare the areas of ABD and ABE if $AB = 4$ and $BC = 9$.

Problem 4. Triangle ABC is inscribed a circle and has an inradius of 4. $AB = 13, AC = 14, BC = 15$. BY bisects arc $\widehat{AC}$ and intersects AC at D. What is the area of BCD?

Problem 5. Right triangle ABC has right angle C and $CB > CA$. Given that D is on BC so $\angle CAD$ is twice $\angle DAB$ and $AC/AD = 2/3$, what is CD/DB?

Problem 6. ABC is a triangle with medians AD, BE, CF. FH is parallel and equal to BE, and FE extended meets CH at G. Show the following:

a. $BFHE$ is a parallelogram.

b. $CH = AD$

c. FG is a median of $\triangle CFH$

d. $FG = \frac{3}{4}BC$

e. $[CFH] = \frac{3}{4}[ABC]$

4 Areas of Triangles

- $\frac{1}{2}bh$
- rs
- $\frac{1}{2}ab\sin C$
- $\sqrt{s(s-a)(s-b)(s-c)}$
- $\frac{abc}{4R}$

Problem 1. Triangle ABC has $AB = 9$, $BC = 10$, and $CD = 11$. A line DE is drawn with D on AB and E and AC such that $DE||BC$ and DE passes through ABC's incenter. Find the length of DE.

Problem 2. The area of triangle ABC is the product of the altitude and median to BC. If $BC = 5$ and the altitude is $\frac{12}{5}$, what is the perimeter of triangle ABC?

Problem 3. Determine a formula for the area of a triangle in terms of its circumradius and angles.

5 Quadrilaterals

Types of Quadrilaterals

- **trapezoid** - one pair of parallel bases connected by legs
- **parallelogram** - two pairs of parallel bases
- **rhombus, rectangle, square**

Cyclic Quadrilaterals

- quadrilateral inscribed in a circle
- opposite angles sum to 180°
- draw in diagonals to get numerous equal angles and similar triangles
- **Ptolemy's Theorem**

$$(AB)(CD) + (AD)(BC) = (AC)(BD)$$

Problem 1. Show that the midpoints of any quadrilateral is a parallelogram.

Problem 2. Prove that the sum of the squares of the sides of a parallelogram equals the sum of the squares of its diagonals. Find the median BM of triangle ABC with sides $AB = 5, BC = 7, CA = 9$.

Problem 3. $ABCD$ has diagonals AC and BD that meet at P. Given that $AB = 6, BP = 4, PD = 6, AP = 8$, and $PC = 3$. Find AD.

Problem 4. Parallelogram $ABCD$ with area 10 has $AB = 8$ and $BC = 5$. E, F, G are on AD, AB, BD, respectively, such that $AE = BF = BG = 3$. Find the area of $DEFG$.

Problem 5. A d-degree chord of a certain circle has length 22. A $2d$-degree chord is 20 units longer than a $3d$-degree chord ($d < 120$). Find the length of a $3d$-degree chord.

Problem 6. Equilateral triangle ABC has D on BC. AD is extended to E such that $\angle EAC = \angle EBC$. Find the length AE.