e and the Complex Numbers: Notes for Day 4

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1 Taylor Series

For a function $f(x)$, its Taylor series can be thought of as a polynomial (possibly of infinitely many terms) that approximates it for x within some interval. It can be constructed by choosing a point and stipulating that f and its Taylor series have the same derivatives at that point.

1.1 Convergence

Having infinitely many terms raises questions about whether polynomials constructed this way give meaningful answers. For example, consider the natural logarithm's Taylor series around $x = 1$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

If we were to pick $x = 2$, successive terms would get bigger and bigger without bound, and this would show no sign of actually producing a finite value.

We'll say a Taylor series "converges" for a given value of x if, by including successive terms, we obtain a sequence of polymomials whose values at x approach some finite number. To illustrate, the Taylor series for e^x around $x = 0$ takes this form:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

If we let x be fixed, we can find some term $\dfrac{x^n}{n!}$ where $n > |x|$. The sum of that and all following terms can be compared to a geometric sum with common ratio $\frac{|x|}{n+1}$:

$$\left|\frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \cdots\right| < \frac{|x|^n}{n!}\left(1 + \frac{|x|}{n+1} + \frac{|x|^2}{(n+1)^2} + \cdots\right)$$

The (finite) value of the geometric sum restricts the growth of the polynomial as successive terms are included. Individual terms tend to zero, ruling out pathological behavior like oscillating indefinitely between two values. So, this Taylor series converges.

1.2 Another Consideration

Convergence isn't enough to make a Taylor series a good approximation for a function. That the polynomial will give finite answers for x within some interval doesn't necessarily mean the answers will match $f(x)$.

Consider Wikipedia's example: $f(x) = 0$ if $x = 0$ and $e^{-\frac{1}{x^2}}$ otherwise. If you take the derivatives of this function at $x = 0$, you'll find they're all zero, which produces merely zero for a Taylor series. Therefore the Taylor series gives a finite answer (zero) everywhere, but the value of the function isn't actually zero everywhere.

Some functions aren't so nasty, and their Taylor approximations do actually give the same answer. To see for ourselves which functions these are, we can use a form of Taylor's theorem.

1.3 Taylor's Theorem and Estimation of Error

Our intent is to derive a polynomial approximating $f(x)$ near $x = a$ and simultaneously find an expression limiting the error between this polynomial and f. We'll assume we can take derivatives of f as many times as we like. We begin with a statement of the Fundamental Theorem of Calculus and apply it recursively to itself:

$$\begin{aligned}
f(x) &= f(a) + \int_a^x f'(t_1)dt_1 \\
&= f(a) + \int_a^x \left(f'(a) + \int_a^{t_1} f''(t_2)dt_2 \right) dt_1 \\
&= f(a) + \int_a^x \left(f'(a) + \int_a^{t_1} \left(f''(a) + \int_a^{t_2} f'''(t_3)dt_3 \right) dt_2 \right) dt_1 \\
&\vdots
\end{aligned}$$

If we split this into a sum of terms, each $f^{(k)}(a)$ ends up in an expression like:

$$\int_a^x \int_a^{t_1} \cdots \int_a^{t_{k-1}} f^{(k)}(a)dt_k \cdots dt_2dt_1$$

...which evaluates (it can be shown inductively) to $\frac{1}{k!}(x-a)^k f^{(k)}(a)$. Putting these together gives the first terms of the Taylor series of $f(x)$ around a.

The innermost integral (at level n, let's say) will produce the one slightly different expression, which gives the difference between the first n terms of the Taylor series and the actual value of f. We hope this amount of error gets smaller as we include more terms:

$$\int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-1}} f^{(n)}(t_n) dt_n \cdots dt_2 dt_1$$

Considering the limits of integration, we see that $f^{(n)}(t_n)$ never needs to be evaluated outside of the range between a and x. If we let m be the maximum absolute value of $f^{(n)}(t_n)$, we know the magnitude of the error is at most:

$$\int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-1}} m dt_n \cdots dt_2 dt_1$$

This evaluates to $\frac{1}{n!}(x-a)^n m$, similarly to the previous integral.

Summarizing all this, the beginning of a Taylor series for f at a:

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2 f''(a)}{2!} + \cdots + \frac{(x-a)^{n-1} f^{(n-1)}(a)}{(n-1)!}$$

...differs from the actual value of $f(x)$ by no more than $\frac{1}{n!}(x-a)^n m$, where m is the maximum absolute value of $f^{(n)}(t)$ for t between a and x.

1.4 Applying Taylor's Theorem

Suppose $f(x) = e^x$. Since e^x is its own derivative, $m = e^x$. The error in the Taylor series for e^x is no more than $\frac{1}{n!}(x-a)^n e^x$, which tends to zero as n increases (since the factorial growth of $\frac{1}{n!}$ outstrips the exponential growth of $(x-a)^n$). This happens regardless of the values of x and a. We decide from this that e^x is equal to its Taylor series for all real x.

For the trigonometric functions sine and cosine, all their derivatives are also sines and cosines with the same amplitude and frequency. (We need to be working in radians for this to be true.) Since these functions never take on values outside of the range $[-1, 1]$, we're safe letting $m = 1$ for both sine and cosine. Then, the error in their Taylor series is no more than $\frac{1}{n!}(x-a)^n$. Once we check that these series actually converge, we can decide that $\cos x$ and $\sin x$ are equal to their Taylor series for all real x.

1.5 Closing Remarks

At this point, if we're willing to assume that the Taylor series for e^x, $\cos x$, and $\sin x$ are the right way to extend their definitions to the whole complex plane, then we can get to Euler's Formula $e^i\theta = \operatorname{cis}\theta$. Using $x = i\theta$ in the Taylor series for e^x, we can recognize the Taylor series for $\cos x$ and $\sin x$ within the resulting expression.

So far we've just shown that these Taylor series work within the real numbers. There's a more roundabout way to arrive at this result: after extending the definitions of these functions to the whole complex plane, one can show that they're equal to their Taylor series all over the complex plane. Since the real numbers are a subfield of the complex numbers, this must hold over all of the real line as well.