1 Complex Numbers

1.1 Definition of a complex number

A complex number takes the form \( z = a + bi \), where \( a, b \) are real numbers and \( i \) is a constant that satisfies \( i^2 + 1 = 0 \). We call \( a \) the real part of \( z \) and \( b \) the imaginary or complex part of \( z \), and write

\[
a = \Re(z),\ b = \Im(z)
\]

The set of complex numbers is denoted by \( \mathbb{C} \); through the bijection \( a + bi \rightarrow (a, b) \), we can identify \( \mathbb{C} \) with \( \mathbb{R}^2 \), the Cartesian plane. Thus each complex number can be represented by a point on the Cartesian plane.

The rules for adding and multiplying extends naturally to complex numbers by using the usual rules of arithmetic and the fact \( i^2 = -1 \). In particular, if \( z_1 = a + bi,\ z_2 = c + di \), then

\[
\begin{align*}
z_1 + z_2 &= (a + c) + (b + d)i \\
z_1 - z_2 &= (a - c) + (b - d)i \\
z_1 z_2 &= (a + bi)(c + di) = (ac - bd) + (ad + bc)i
\end{align*}
\]

However, we cannot compare two complex numbers by inequality unless they are both real.

1.2 Modulus and Conjugate

Two additional notions are especially useful when studying complex numbers. The modulus or absolute value of \( z = a + bi \) is defined as

\[
|z| = \sqrt{a^2 + b^2}
\]
If \( z_1 \) and \( z_2 \) are complex numbers, with \( A \) and \( B \) being their corresponding points on \( \mathbb{R}^2 \), then by Pythagoras’ Theorem \(|z_1 - z_2|\) is precisely the distance between \( A \) and \( B \). Thus, using the triangle inequality on the points \( O = (0, 0) \), \( z_1 \) and \( z_1 + z_2 \),

\[ |z_1 + z_2| \leq |z_1| + |z_2| \]

Furthermore, by direct computation we can know that

\[ |z_1 z_2| = |z_1||z_2| \]

The conjugate of a complex number \( z = a + bi \) is defined as \( \bar{z} = a - bi \). Conjugation is preserved under addition and multiplication:

\[ \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \]
\[ \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \]

Furthermore,

\[ z \bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2 \]

Thus

\[ \frac{1}{z} = \frac{\bar{z}}{|z|^2} \]

On the Cartesian plane, the conjugate of a complex number is its reflection across the \( x \)-axis.

### 1.3 Polar Form

Given a complex number \( z \). If we let \( r = |z| \), then \( c+di = \frac{1}{r} z \) satisfies \( c^2+d^2 = 1 \). Thus \((c, d)\) lies on the unit circle, and there exists a unique \( \theta \) up to multiples of \( 2\pi \) such \( c = \cos(\theta) \), \( d = \sin(\theta) \). Thus,

\[ z = r(\cos(\theta) + \sin(\theta)i) \]

We call this the polar form of \( z \), and \( \theta \) is called the argument. On the Cartesian plane, \((r, \theta)\) is just the polar coordinate of \( z \). As we will soon see, we can also write

\[ z = re^{i\theta} \]

The main advantage of the polar form is that multiplication becomes much easier. In fact, we have

\[ z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)} \]

In other words, to multiply two complex numbers, one simply multiplies their modulus and adds their arguments.
As an exercise, find the real part, complex part, modulus and conjugate of $2 + i$. Also compute $(1 + i)^{100}$.

Answer: $2, 1, \sqrt{5}, 2 - i$

$$(1 + i)^{100} = (\sqrt{2}(\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4})))^{100} = -2^{50}$$
2 Facts From Calculus

We will use the notions of limit, derivative and integrals. If you have never heard of these definitions, it would be better to learn them from a Calculus Textbook. If you already know what these means, the following is a quick review of their key properties.

2.1 Limits

Definition 1 We say a series of complex numbers \( a_1, a_2, \cdots \) converges to a complex number \( A \), if for any \( \epsilon > 0 \), there exists an \( N \) such that for any \( n > N \), \( |a_i - A| \leq \epsilon \). We can also write \( \lim_{i \to \infty} a_i = A \). If no such \( A \) exists, we say \( a_i \) diverges.

For an infinite sum \( \sum_{i=1}^{\infty} a_i \), we say it converges to \( A \), or \( \sum_{i=1}^{\infty} a_i = A \), if the partial sums \( b_n = \sum_{i=1}^{n} a_i \) \((n = 1, 2, \cdots)\) converges to \( A \). Similarly, we say \( \prod_{i=1}^{\infty} a_i = A \) if the partial products \( b_n = \prod_{i=1}^{n} a_i \) \((n = 1, 2, \cdots)\) converges to \( A \).

The following rule by Cauchy is especially useful when testing for converges,

Property 1 (Cauchy) A series of complex numbers \( a_1, a_2, \cdots \) converges to a complex number if and only if the following criterion holds:

For any \( \epsilon > 0 \) there exists an \( N \) such that for any \( m, n > N \), \( |a_m - a_n| \leq \epsilon \)

Similarly, an infinite sum \( \sum_{i=1}^{\infty} a_i \) converges to some complex number if and only if the following criterion holds:

For any \( \epsilon > 0 \) there exists an \( N \) such that for any \( m, n > N \), \( |\sum_{i=m}^{n} a_i| \leq \epsilon \).

Many rules can be derived from this property, including

- The Comparison Rule: for a series of complex numbers of \( a_i \) and a series of positive reals \( b_i \), if \( \sum_{i=1}^{\infty} b_i \) converges and \( |a_i| \leq b_i \) holds for all \( i \), then \( \sum_{i=1}^{\infty} a_i \) converges.
- The Converge-to-zero Rule: if \( \sum_{i=1}^{\infty} a_i \) converges, then \( \lim_{i \to \infty} a_i = 0 \).

You might be less familiar with the notion of convergence for products. For the sake of this course, the following rule will be sufficient.
**Property 2** If \( \sum_{i=1}^{\infty} |a_i| \) converges, then \( \prod_{i=1}^{\infty} (1 + a_i) \) converges.

As an exercise, show that \( \prod_{n=1}^{\infty} (1 - \frac{1}{(n+1)^2}) \) converges and find its exact value.

Answer: \( \frac{1}{2} \)

### 2.2 Derivatives and Integrals

The **Derivative at** \( x_0 \) of a function \( f : \mathbb{R} \to \mathbb{C} \) is the limit

\[
f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}
\]

We say \( f \) is **differentiable at** \( x_0 \) if this limit exists. The following are useful rules when calculating derivatives:

\[
\begin{align*}
(x^n)' &= nx^{n-1}, \quad (\ln(|x|))' = \frac{1}{x}, \quad (e^x)' = e^x \\
\sin'(x) &= \cos(x), \quad \cos'(x) = -\sin(x) \\
(f + g)' &= f' + g' \\
(fg)' &= g'f + f'g \\
\left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2} \\
(f \circ g)' &= (f' \circ g) \cdot g'
\end{align*}
\]

As an exercise, find the derivative of \( f(x) = e^{\sin \cos(x)} \).

The **integral on** \( [a, b] \) \( (a < b) \) of a continuous function \( f \) is defined as

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(a + \frac{(b-a)i}{n})
\]

It can be shown that the limit always exists. The Fundamental Theorem of Calculus is the most basic tool for finding integrals. It states that

\[
\int_a^b f'(x) \, dx = f(b) - f(a)
\]

There are numerous cool tricks on finding integrals, including integration by part, change of variables, and finding anti-derivatives directly. One of the coolest integrals I have seen is the following:
Challenge Problem 1 Find
\[ \int_0^\pi \ln \sin(x) \, dx = \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi - \epsilon} \ln \sin(x) \, dx \]
And give justification.

If a bound of an integral is \( \pm \infty \), then the integral is defined by replacing that bound by \( N \) and taking the limit as \( N \to \pm \infty \). For example,
\[ \int_1^\infty \frac{1}{x^2} \, dx = \lim_{N \to \infty} \int_1^N \frac{1}{x^2} \, dx = \lim_{N \to \infty} (1 - \frac{1}{N}) = 1 \]

As an exercise, find the following integrals
\[ \int_0^1 \frac{1}{x + 1} \, dx \]
\[ \int_0^1 \sin(x) \, dx \]
\[ \int_0^1 \tan(x) \, dx \]
Answers: \( \ln(2) \), \( 1 - \cos(1) \), \( -\ln \cos(1) \)

2.3 Taylor Expansion

Taylor Expansion is a remarkable formula that enables us to treat an arbitrary function \( f \) as polynomials; the basic idea is to construct a polynomial \( p \) that “almost agrees” with \( f \).

Definition 2 The \( n^{th} \) derivative of \( f \), \( f^{(n)} \), is defined recursively as
\[ f^{(0)} = f, f^{(n)} = (f^{(n-1)})' \]
If \( f^{(n)}(a) \) is well-defined for all \( n \geq 0 \), then we call \( f \) smooth at \( a \). The Taylor series of a smooth \( f : \mathbb{R} \to \mathbb{C} \) at a point \( a \) is the infinite sum
\[ p(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)(x - a)^n}{n!} \]
For this class, we can assume that all functions are smooth wherever they are defined. For most functions that we will be concerned with, we have the Taylor Expansion identity:
\[ f(x) = p(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)(x - a)^n}{n!} \]
Furthermore, the rules of derivatives and integration also applies. In particular,

\[ f'(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x - a)^{n-1}}{(n-1)!} \]

\[ \int f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x - a)^{n+1}}{(n+1)!} + C \]

If we take \( a = 0 \), the resulting expansion is called the **MacLaurian Series**.

\[ f(x) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)x^n}{n!} \]

As an exercise, find the Taylor expansion of \( f(x) = e^x \) at \( a = 0 \), and plug it into the two formulas above.

Answer: All three give \( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \)