S15641: Relativity Crash Course Lecture 1

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1 Introduction: What is relativity?

Relativity is the physics which describes the dynamics of observers in relative motion. More generally, relativity is the study of space and time. In physics, relativity refers to two groundbreaking theories by Einstein:

- Special Theory of Relativity: Published in 1905, it was necessary to describe the physics of observers moving at relative speeds close to the speed of light.
- General Theory of Relativity: Published in 1915, it makes gravity compatible with special relativity by describing gravity as the curvature of space time instead of as a force.

2 The Principle of Relativity

In 1632, Galileo proposed a thought experiment in his book *Dialogue Concerning the Two Chief World Systems* about the orbits of planets.

Figure 1: Galileo's ship

"Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it.

With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need to throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction.

When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in

> the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. And if smoke is made by burning some incense, it will be seen going up in the form of a little cloud, remaining still and moving no more toward one side than the other.
>
> The cause of all these correspondences of effects is the fact that the ship's motion is common to all the things contained in it, and to the air also. That is why I said you should be below decks; for if this took place above in the open air, which would not follow the course of the ship, more or less noticeable differences would be seen in some of the effects noted."

All this to convey the idea that *all physics looks the same in a ship at rest, and one moving at a constant speed.* This is formalized in the **Principle of Relativity** which states that there is no physical difference between inertial reference frames.

3 Galilean Relativity

A **reference frame** is a coordinate system used by some observer to measure times and distances (in which the observer sits at the spatial origin of the coordinate system). In relativity, we draw a reference frame like this:

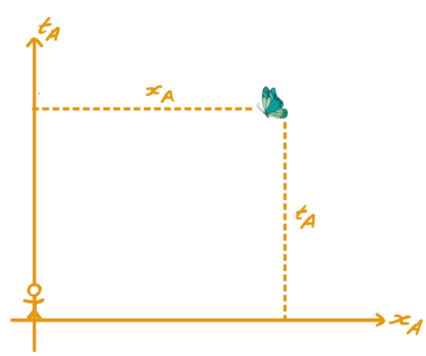

Figure 2: Alice's Reference frame in which $t_A = 0$ is the time when Alice starts her clock, and she remains at position $x_A = 0$

Note that unlike you may have seen in other physics classes, in relativity, we usually draw time on the vertical axis, and space on the horizontal axis. In this reference frame, the coordinates of the butterfly are (t_A, x_A). Coordinates give a label to an *event*, a moment in time and space.

Different coordinate systems can label the same event with different coordinates. For example, consider a different reference frame, which is translated with respect to the first.

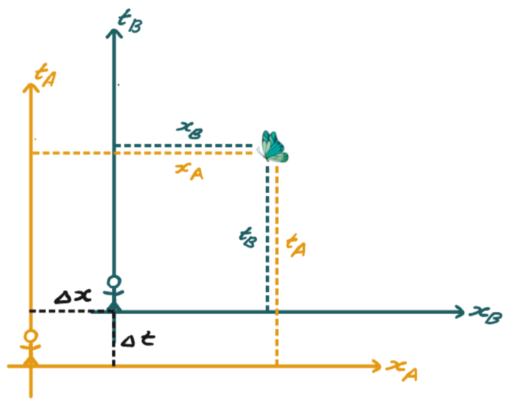

Figure 3: A new reference frame. (Bob's frame) which is translated with respect to Alice. According to Alice, Bob stands at position Δx and starts his clock at Δt

The coordinates they measure for the butterfly are related by:

$$x_B = x_A - \Delta x \tag{1}$$

$$t_B = t_A - \Delta t \tag{2}$$

To keep things simple, we always choose different reference frames to have the same *origin*. This means that two different observers sync their clocks to 0 when they are at the same position.

A **inertial reference frame** is a frame which is moving at a constant speed (which can be zero). Consider two different inertial frames:

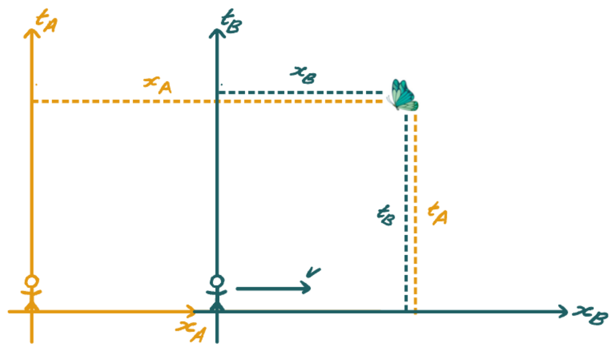

Figure 4: A new reference frame. (Bob's frame) where Bob is moving at speed v to the right according to Alice. The two coordinate systems have the same origin so at $t_A = t_B = 0$, both Alice and Bob are at $x_A = x_B = 0$

How will their measurements of the butterfly's position differ? According to Alice, the coordinates of the butterfly are still (t_A, x_A).
However, since Bob is moving to the right, he sees the butterfly as moving toward him. The butterfly's position will be equal to x_A *minus the distance that Bob moved forward*.

$$x_B = x_A - vt_A \tag{3}$$

$$t_B = t_A \tag{4}$$

If we want, we can write this transformation law in terms of matrices:

$$\begin{bmatrix} t_B \\ x_B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -v & 1 \end{bmatrix} \begin{bmatrix} t_A \\ x_A \end{bmatrix} \tag{5}$$

The transformation law eq. (5) is called a **Galilean Transformation**. Importantly, in Galilean relativity, observers in different inertial reference frames still measure the same time. eq. (5) allows us to convert Alice's coordinates to Bob's coordinates. The inverse transformation, lets us convert Bob's coordinates to Alice's:

$$\begin{bmatrix} t_A \\ x_A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \begin{bmatrix} t_B \\ x_B \end{bmatrix} \tag{6}$$

Note that this is compatible with the equivalent view (using the Principle of relativity) that Bob is standing still and *Alice is moving to the left* with speed v (velocity $-v$).
Now suppose Alice sees the butterfly moving with some velocity u_A. That is:

$$u_A = \frac{\Delta x_A}{\Delta t_A} = \frac{x_{A,f} - x_{A,i}}{t_{A,f} - t_{A,i}} \tag{7}$$

How will this compare to the velocity of the butterfly as measured by Bob? Subbing in our Galilean transformation law eq. (6), we get:

$$u_A = \frac{x_{A,f} - x_{A,i}}{t_{A,f} - t_{A,i}} = \frac{x_{B,f} + vt_{B,f} - x_{B,i} - vt_{B,i}}{t_{B,f} - t_{B,i}} = u_B + v$$

This gives us the *velocity addition formula*:

$$u_B = u_A - v \tag{8}$$

So if Bob is moving the same direction as the butterfly, he will see it move slower than Alice, and if he is moving the opposite direction, he will see it move faster. You may have noticed this effect when you are in a car on the highway. The cars going the same direction as you seem slow, but the cars moving opposite you seem really fast.

Now let's see how acceleration compares in the two frames:

$$a_A = \frac{u_{A,f} - u_{A,i}}{t_{A,f} - t_{A,i}} = \frac{u_{B,f} + v - u_{B,i} - v}{t_{B,f} - t_{B,i}} = a_B$$

So the two reference frames measure the same acceleration!

$$a_A = a_B \tag{9}$$

eq. (9) can be understood as a mathematical restatement of the principle of relativity. Because accelerations are the same, Newton's second law $F = ma$ remains intact is all frames. The same forces are always measured, same work required, etc. *All dynamics are the same.*

4 Electromagnetism

Although Galilean relativity might feel intuitive, issues arise when we take it to extremes.
Electromagnetism is the study of how electric charges interact with each other. In 1865, James Clerk Maxwell published a paper titled "A Dynamical Theory of the Electromagnetic Field" which attempted to explain these interactions. The crux of his theory came from the compilation of four equations now know as **Maxwell's Equations**:

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{B} &= \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{aligned}$$

Using these equations, Maxwell was able to model electromagnetic waves. These are solutions to Maxwell's equations with no sources of charge ($\rho = 0$ and $\vec{j} = 0$). Maxwell derived that these electromagnetic waves moved at a fixed speed of $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 2.998 \cdot 10^8 \frac{m}{s}$. Amazingly, this speed was extremely close to the experimentally measured speed of light! Thus he made the realization that electromagnetic waves precisely modeled light.

And so we have a theoretical prediction for the speed of light being $c = 2.998 \cdot 10^8 \frac{m}{s}$! But ... in what reference frame? The velocity addition formula suggests that someone moving alongside a beam of light should see it moving slower. Maxwell's equations didn't single out a specific reference frame, but how could something move at the same speed for everyone?

5 The Luminiferous Ether

To try to solve this problem, in the late 1800s,the **luminiferous ether** was proposed as the medium for light. Similar to how ocean waves propagate through water and sound waves propagate through the air, it was proposed that light waves propagate through the ether.

This way one can say that the speed of light is c *in the ether's reference frame*. This parallels the way the speed of sound is $343 \frac{m}{s}$ in the air's reference frame (in which the air is stationary). The existence of the ether would solve our problem from before, because it would pick out a special reference frame for light to travel at the speed c predicted by Maxwell's Equations.

In 1887, Albert A. Michelson and Edward W. Morley set out to measure the motion of the earth relative to the ether.

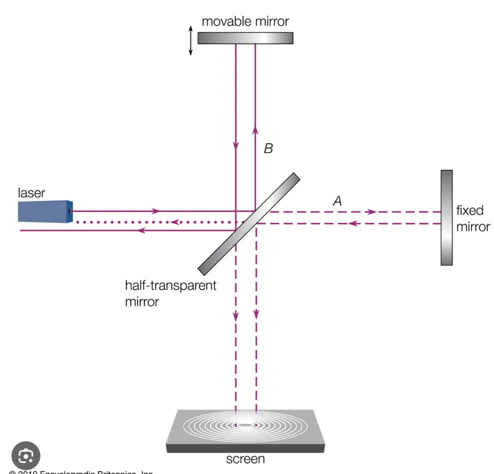

Figure 5: The interferometry setup used for the Michelson-Morley experiment

The diagram above (Figure 7) shows the general setup of the Michelson Morley experiment, called an **interferometer**. There are a few key components to describe here. First a light wave is produced from the laser on the left. This light then hits a half-transparent mirror (also known as a beam-splitter) which partially reflects the light wave and partially transmits it. This directs it into two different branches: branch A and branch B. Within each branch, the light then is reflected off of a mirror and detected by a screen at the bottom of the setup.

Recall that the assumption under this experiment is that there exists an ether which moves at some velocity, v, with respect to the earth. Thus, the light will propogate with different speeds depending on how its direction is aligned with the motion of the ether. After going through the calculation of the time difference (see exercise below!) we will see that it depends on the velocity of the ether with respect to the earth. By measuring the time difference, we can get this velocity. This is precisely what we want!

So Michelson and Morley did exactly this. And they found that this velocity is $v = 0$. So the ether exactly aligned with the reference frame of the earth, the eart already happens to be in the exact reference frame where the speed of light is $c = 3 \cdot 10^8 \frac{m}{s}$. This seems unlikely, but there are no theoretical issues with this. The issue comes in when you remember that the earth orbits around the sun. This means that the earth's velocity with respect to the ether must change as the earth continues its orbit. Unfortunately, Michelson and Morley got $v = 0$ at all times in the year. This however makes no sense! The only resolution here is that the ether doesn't exist.

6 The Postulates of Relativity

So what now? We know that the principle of relativity, Galilean relativity, and Maxwell's equations can't all hold. Thus one of these can't be true in general. When Einstein introduced special relativity, he proposed that Galilean relativity was the postulate that should be tossed. Thus the two postulates of special relativity are

1. The principle of relativity.

2. The speed of light, c, is constant in all reference frames.

7 Exercises

Problem 1: Galilean Transformations

(a) Suppose Alice is standing at the front of a train car of length L moving with speed v in the $+x$ direction, while Bob waits at the train station. Alice rolls a ball to the back of the train car with speed u. How long does it take for the ball to reach the back of the train car according to Alice?

What about according to Bob? How far did the ball travel according to each of them?

(b) A rocket ship is travelling in outer space and sees a purple planet moving toward them with constant speed v. They also see an alien ship moving away from them with speed u. How does the alien ship observe the planet?

Problem 2. Michelson and Morley Consider two identical swimmers Michelson and Morley, who in a static pool swim with speed u. Suppose they decide to have a race at the Ether river which flows West to East with velocity v (in ground reference frame). Michelson decides to swim across the river, which has width L, and then back. Morley, on the other hand, decides to swim the same length L downstream, and then back upstream. Both swimmers travel a total distance of $2L$. But will they take the same time?

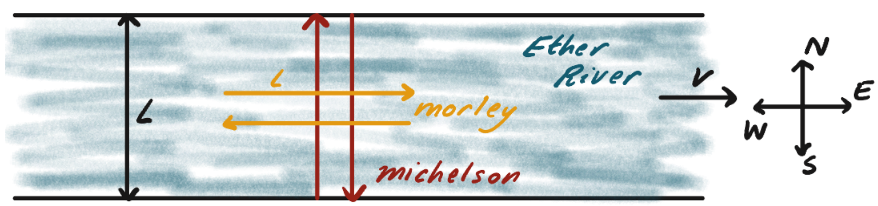

Figure 6: The two swimmers and the river *in the ground reference frame.* Remember, the swimmers swim with speed u only in still water

(a) (Morley): How fast does Morley swim when going downstream? What about upstream? (all in ground reference frame). What is the total trip time?

(b) (Michelson): With what angle does Michelson have to swim to fight the river and make sure he swims straight across (no velocity in East/West directions in ground frame). What is Michelson's speed in the vertical direction only? What is his total trip time?

(c) Who wins the race? And by how much?

(d) Draw a picture depicting this race in the river's reference frame instead of the ground reference frame.

Problem 3: How fast can you go? Suppose you have a long line of people lined up from left to right. Call the person at the left end of the line Alice and the person at the right end of the line Bob. Each person (except for Alice and Bob) has a neighbor to their left and a neighbor to their right. In each person's reference frame, they observe their left neighbor to be moving to the left at a speed of $100\frac{m}{s}$ and their right neighbor to be moving to the right at a speed of $100\frac{m}{s}$.

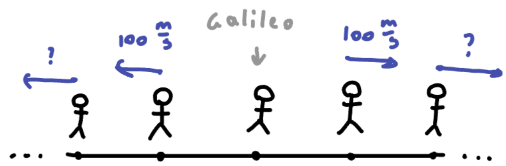

Figure 7: A line of people moving as described in the problem. Note the drawn velocities (in blue) are in the reference frame of Galileo.

(a) Suppose there are N people in this line. How fast does Alice observe Bob is moving? (Assume the velocity addition formula derived above holds here)

(b) How many people would there need to be for Alice to observe Bob to be moving faster than $c = 3 \cdot 10^8 \frac{m}{s}$.

(c) Now suppose Alice did observe Bob to be moving faster than $c = 3 \cdot 10^8 \frac{m}{s}$. Does this pose any issues (knowing that the speed of light is constant in all reference frames)?
Hint: Imagine a race between Bob and a beam of light. Who would win according to Alice? Who would win according to Bob?

This further suggests that the velocity addition formula we derived is faulty. We will soon find that special relativity gives rise to a different velocity addition formula.