A vector space \( V \) over a field \( F \) has the operations of addition and scalar multiplication, and satisfies several basic laws. A vector space in a vector space is a subspace.

A vector \( v \in V \) is a linear combination of vectors of \( S \subseteq V \) if there exist a finite number of vectors \( u_1, u_2, \ldots u_n \in S \) and scalars \( a_1, a_2, \ldots a_n \in F \) such that

\[
v = a_1 u_1 + \cdots + a_n u_n.
\]

If 0 can be nontrivially written in this form, \( S \) is linearly dependent. The set of all \( v \) in the above form is the subspace generated (spanned) by \( S \).

A basis \( \beta \) for \( V \) is a linearly independent subset of \( V \) that generates \( V \).

Replacement Theorem: (Simplified) Every linearly independent set can be made into a basis by adding elements.

Every basis for \( V \) contains the same number of vectors. The unique number of vectors in each basis is the dimension of \( V \) (\( \dim(V) \)).

Every vector space has a basis.
For vector spaces $V$ and $W$ over $F$, a function $T: V \to W$ is a linear transformation (homomorphism) if for all $x, y \in V$ and $c \in F$,

(a) $T(x + y) = T(x) + T(y)$

(b) $T(cx) = cT(x)$

The **null space** or kernel is the set of all vectors $x$ in $V$ such that $T(x) = 0$.

$$N(T) = \{x \in V | T(x) = 0\}$$

The **range** or image is the subset of $W$ consisting of all images of vectors in $V$.

$$R(T) = \{T(x) | x \in V\}$$

Both are subspaces. **nullity**$(T)$ and **rank**$(T)$ denote the dimensions of $N(T)$ and $R(T)$, respectively.

**Dimension Theorem**: If $V$ is finite-dimensional, $\text{nullity}(T) + \text{rank}(T) = \text{dim}(V)$

*Linear transformations (over finite-dimensional vector spaces) can be viewed as left-multiplication by matrices, so linear transformations under composition and their corresponding matrices under multiplication follow the same laws. This is a motivating factor for the definition of matrix multiplication.* Facts about matrices can be proved by using linear transformations, or vice versa.

**Matrix product**:

Let $A$ be a $m \times n$ and $B$ be a $n \times p$ matrix. The product $AB$ is the $m \times p$ matrix with entries

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}, \ 1 \leq i \leq m, \ 1 \leq j \leq p$$

**Interpretation of the product $AB$**:

1. **Row picture**: Each row of $A$ multiplies the whole matrix $B$.
2. **Column picture**: $A$ is multiplied by each column of $B$. Each column of $AB$ is a linear combination of the columns of $A$, with the coefficients of the linear combination being the entries in the column of $B$.
3. **Row-column picture**: $C_{ij}$ is the dot product of row $i$ of $A$ and column $j$ of $B$. 
The matrix representation of $T$ in $\beta = \{v_1, ..., v_n\}$ and $\gamma$ is $A = [T]_\beta^\gamma$. Load the coordinates of $T(v_i)$ into the $i$th column. $[I_V]_\beta^\gamma$ changes $\beta$-coordinates to $\gamma$-coordinates. So:

$$[T]_\gamma = [I_V]_\gamma^\beta [T]_\beta^\gamma$$

$$B = QAQ^{-1}$$

<table>
<thead>
<tr>
<th>Linear transformations $T, U$</th>
<th>Matrices $A, B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>rank$(TU) \leq \min$rank$(T), \text{rank}(U))$</td>
<td>rank$(AB) \leq \min$rank$(A), \text{rank}(B))$</td>
</tr>
</tbody>
</table>

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**Fundamental Theorem of Linear Algebra (Part 1):**

Dimensions of the Four Subspaces: $A$ is $m \times n$, rank$(A)=r$ (If the field is complex, replace $A^T$ by $A^*$.)

- **Row space** $C(A^T)$
  - $\{A^T y\}$
  - Dimension $r$

- **Column space** $C(A)$
  - $\{Ax\}$
  - Dimension $r$

- **Nullspace** $N(A)$
  - $\{x|Ax = 0\}$
  - Dimension $n-r$

- **Left nullspace** $N(A^T)$
  - $\{y|A^T y = 0\}$
  - Dimension $m-r$

$$F^n = C(A)^T \oplus N(A)$$

$$F^m = C(A) \oplus N(A^T)$$
The **determinant** (denoted $|A|$ or $\det(A)$) is a function from the set of square matrices to the field $F$, satisfying the following conditions:

1. The determinant of the nxn identity matrix is 1, i.e. $\det(I) = 1$.
2. If two rows of $A$ are equal, then $\det(A) = 0$, i.e. the determinant is alternating.
3. The determinant is a linear function of each row separately, i.e. it is $n$-linear. That is, if $a_1, \ldots a_n, u, v$ are rows with $n$ elements,

$$\det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k \det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

*These properties completely characterize the determinant.*

**Cofactor Expansion**: Recursive, useful with many zeros, perhaps with induction.

(Row)

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det(M_{ij})$$

(Column)

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det(M_{ij})$$

where $M_{ij}$ is $A$ with the $i$th row and $j$th column removed.

**Cramer’s Rule**:

If $A$ is a nxn matrix and $\det(A) \neq 0$ then $Ax = b$ has the unique solution given by

$$x_i = \frac{\det(B_i)}{\det(A)}, 1 \leq i \leq n$$

Where $B_i$ is $A$ with the $i$th column replaced by $b$. If $\det(A) = 0$, then $A$ is singular (has no inverse).
Let T be a linear operator (or matrix) on V. A nonzero vector \( v \in V \) is an **eigenvector** of T if there exists a scalar \( \lambda \), called the **eigenvalue**, such that \( T(v) = \lambda v \). The **eigenspace** of \( \lambda \) is the set of all eigenvectors corresponding to \( \lambda \): \( E_\lambda = \{ x \in V | T(x) = \lambda x \} \).

The **characteristic polynomial** of a matrix A is \( \det(A - \lambda I) \). The zeros of the polynomial are the eigenvalues of A. For each eigenvalue solve \( Av = \lambda v \) to find linearly independent eigenvalues that span the eigenspace.

If there are \( n \) linearly independent eigenvalues, T (A) is diagonalizable:

\[
[T]_\gamma = [I_V]_\beta^T [T]_\beta [I_V]^T_\beta
\]

\[
A = Q\Lambda Q^{-1}
\]

Where \( \Lambda = [T]_\beta \) is a diagonal matrix.

Applications to recursive sequences, probability (Markov chains).

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The **incidence matrix** of a graph: A has a row and column for each vertex, and \( A_{ij} = 1 \) if vertices i and j are connected by an edge, and 0 otherwise.

The incidence matrix A for a family of subsets \( \{S_1, ..., S_n\} \) containing elements \( \{x_1, ..., x_m\} \) has \( A_{ij} = \begin{cases} 1 & \text{if } x_i \in S_j \\ 0 & \text{if } x_i \notin S_j \end{cases} \). Exploring \( AA^T \) and using properties of ranks, determinants, linear dependency, etc. may give conclusions about the sets. Working in the field \( \mathbb{Z}_2 \) on problems dealing with parity may help.